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THE UNIVERSITY OF TOLEDO
Comprehensive Examination (Analysis)
Spring 2005
M.S. Applied Mathematics

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Solve any four questions from each of the parts A & B. 100% will be awarded for complete answers to four questions in each part. **To obtain full credit you must show all of your work**. *Indicate clearly which eight problems you wish to have graded.*

PART A: REAL ANALYSIS

1. i) State the definition of convergence for a sequence of real numbers.

   ii) Prove that if $\lim_{n\to\infty} a_n = a$ and k is any real number then $\lim_{n\to\infty} ka_n = ka$.

   iii) If $a_0 = 0$ and $a_{n+1} = \frac{1+2a_n}{2+a_n}$ prove that the sequence $\{a_n\}$ is convergent and find the limit. State carefully any results that you quote in your proof.

2. i) State the definition of continuity for a real-valued function whose domain is a subset of $\mathbb{R}$.

   ii) Prove from the definition in (i) that the function $f(x) = x^2$ is continuous at $x = 2$.

   iii) Suppose that $g(x)$ is a twice-differentiable odd function, that is, $g(-x) = -g(x)$ defined on an open interval of $\mathbb{R}$ that contains 0. Find $g''(0)$ and justify your answer.

   [Hint: What do you think can be said about $g'(x)$? Look at some examples.]

3. i) Prove that a convergent sequence of real numbers is bounded.

   ii) Suppose $\lim_{n\to\infty} a_n = \ell$ and $\lim_{n\to\infty} b_n = \ell$ and for each positive integer n, $a_n \le c_n \le b_n$. Prove $\lim_{n\to\infty} c_n = \ell$.

4. i) State Rolle's Theorem.

   ii) Examine whether the hypotheses and conclusions of Rolle's Theorem hold for $f(x)$ defined by $x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$ on the interval $\left[\frac{-1}{2\pi}, \frac{1}{2\pi}\right]$.

   iii) If $a_0, a_1, \cdots, a_n$ are real numbers such that $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0$ prove that the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$

   has a solution between 0 and 1.

5. i) Prove that if f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all $x \in [a, b]$ then there exists μ such that $m \leq \mu \leq M$ and $\int_a^b f = \mu(b - a)$.

   ii) Prove that if f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

   [Hint: One way is to use the facts that f is bounded and uniformly continuous on $[a, b]$.]

6. i) Let (X, d) be a metric space and let $Y \subset X$. Show that a subset $A \subset Y$ is open in Y if and only if there exists a subset B open in X such that $A = B \cap Y$.

   ii) Let (X, d) and (Y, d') be metric spaces. Prove that a function $f(x)$ is continuous at $x \in X$ if and only if for all sequences $\{x_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} f(x_n) = f(x)$.

PART B: COMPLEX ANALYSIS

1. Derive:

   a) The Taylor series expansion of $f(z) = \frac{1}{1-z}$ about the point $z = i$ and indicate the radius of convergence.

   b) The Laurent series expansion of $f(z) = \frac{1}{(1-z)(3-z)}$ for the domain $1 < |z| < 3$.

2. Let $f(z) = \begin{cases} \frac{(\overline{z})^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

   a) Show that $f'(0)$ does not exist.

   b) Let u and v denote the real and imaginary components of f. Show that u and v satisfy the Cauchy-Riemann equations at $z = 0$.

3. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx$.

4. Find the image of:

   a) the vertical line $x = a$, $a > 0$ under the map $f(z) = z^2 + 1$. Draw the image in the uv-plane and indicate the orientation.

   b) the infinite strip $\frac{\pi}{4} \leq y \leq \frac{\pi}{2}$ under the map $f(z) = e^{2z}$.

5. Evaluate the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\, dx.$$

6. i) Let $f(z) = u(x, y) + iv(x, y)$ be analytic in some domain D in $\mathbb{C}$. Show that u and v satisfy the Cauchy-Riemann equations in D.

   ii) Find an analytic function on $\mathbb{C}$ whose real part is

      a) $x^3 - 3xy^2$

      b) $\exp(x) \cos y$