Probability and Statistical Theory

MS Comprehensive Examination

April 10, 2004

Instructions:

Please answer all three questions.
Point Values: 50, 25, 25

Record your answers in your blue books.

Show all of your computations.
Prove all of your assertions or quote the appropriate theorems.
Books, notes, and calculators *may be used*.

You have three hours.

Comprehensive Examination April 10, 2004

1. Suppose $\vec{X} = \{X_1 \cdots, X_n\}$ $(n \geq 3)$ are independent samples from

$$f(x|\theta) = \theta e^{-\theta x}, x \geq 0, \theta > 0.$$

a. Let $Y = F(X)$, where $F(.)$ is the cdf of $f(.)$. Show that $Y \sim U(0,1)$.

b. Let $Y_1 = F(X_1)$ and $Y_2 = F(X_2)$. Find $\mathrm{P}\left(\frac{1}{4} \leq Y_1 + Y_2 \leq \frac{1}{2}\right)$.

c. Show that $T = \sum_{i=1}^{n} X_{(i)}$ is a complete sufficient statistic for θ, where $X_{(i)}$ is the ith order statistic.

d. Show that $W = \dfrac{X_{(1)}}{\sum_{i=1}^{n} X_{(i)}}$ is an ancillary statistic.

e. Find $\mathrm{E}[W]$.

f. Find the MLE $\hat{\theta}$ for θ.

g. Find the UMVUE $\tilde{\theta}$ for θ.

h. Does $\tilde{\theta}$ attain the C-R Lower Bound?

i. Find a UMP level α test $\phi(\vec{x})$ for $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$.

2. Let X_1 and X_2 denote a random sample of size 2 from the uniform distribution over the interval (0,b), U(0,b).

a. Write the likelihood function for this data as a function of "b".
b. Find the maximum likelihood estimator for "b".
c. Find the CDF, pdf, expectation and standard deviation of the estimator in part b.
d. For testing H_0:b=1 versus H_A:b$\neq$ 1 at level of significance α, write the likelihood ratio λ as a function of the data. Sketch λ as a function of the maximum likelihood estimator from part b.
e. Find the likelihood ratio test, i.e. the test statistic (if you choose one other than λ), the critical value, as a function of the level of significance α.
f. Find and sketch the power function of this test.
g. Find an upper 95% confidence interval for "b", of the form $(kX_{(2)}, \infty)$, where "k" is a constant and $X_{(2)}$ denotes the second order statistic.

3.Consider the following experiment: In a sequence of independent trials, roll a fair, four-sided, die (outcomes in {1,2,3,4} all equally likely on each trial). Roll only until the first time a "4" is rolled. Let X denote the sum of the results before that first "4" is rolled. Our aim is to find the expectation and standard deviation of X. The problem consists of some steps along the way, as detailed below.

a. Denote by Y the trial number of the first "4". Name and write down the distribution of Y.
b. Derive the moment generating function $M_Y(t)$ for Y. Be sure to state the domain of this function.
c. Use your answer in part b to find E(Y) and Var(Y). State why the domain of $M_Y(t)$ matters.
d. Let X_i denote the number rolled on trial "i", where i<Y. Write down the conditional distribution, expectation, and variance of X_i given i<Y.
e. Use the formula E(X) = E(E(X|Y)) to find E(X).
f. Use the formula Var(X) = Var (E(X|Y)) + E(Var(X|Y)) to finish the problem by finding the standard deviation of X.