Differential Equations - M.S. Exam

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The Examination Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

M.S. Comprehensive Examination

You should work <u>any three</u> of the four problems on each of the two parts (ODE, PDE). Show all your work and clearly indicate your answers.

PART ODE

1. Find three linearly independent solutions of the equation

$$y'''(t) + y'(t) - 2y(t) = 0$$

   and prove that they are linearly independent.

2. Show that there are no negative eigenvalues for $\phi''(x) = -\lambda\phi(x)$, where $\phi'(0) = 0$ and $\phi'(L) = 0$ on $[0,\ L]$.

3. Consider the system

$$\frac{dx}{dt} = ax + by,\ \frac{dy}{dt} = cx + dy\ (a, b, c, d \text{ real}).$$

   Show that if $ad - bc \neq 0$, the only equilibrium point is $(0,\ 0)$.

   Show that if $ad - bc = 0$, there are an infinite number of equilibrium points. Is $(0,\ 0)$ an "isolated" equilibrium point? Why?

4. Locate the critical points for the nonlinear system

$$\begin{aligned} x'(t) &= x(1-y) \\ y'(t) &= y(1-2x). \end{aligned}$$

   Determine the stability of the system for each of its critical points and sketch the trajectories, all on the same set of axes.

PART PDE

1. Solve the first-order equation with the condition:

$$u_x = 4u_y,\ u(0,\, y) = 8e^{-3y}.$$

2. Using the idea of superposition find two subproblems that are easier to solve than is

$$\begin{aligned} &u_t = u_{xx} + \sin \pi x && 0 < x < 1,\ 0 < t < \infty \\ &u(0,\, t) = u(1,\, t) = 0 && 0 < t < \infty \\ &u(x,\, 0) = \sin 2\pi x && 0 \le x \le 1. \end{aligned}$$

Don't solve just state the two problems!

3. Find the function $u(x,\, t)$ that satisfies the following four conditions:

$$\begin{aligned} &u_t = u_{xx} && 0 < x < 1,\ 0 < t < \infty \\ &u(0,\, t) = u(1,\, t) = 0 && 0 < t < \infty \\ &u(x,\, 0) = 1 && 0 \le x \le 1. \end{aligned}$$

4. The heat equation that we have been studying is the linear one, whether homogeneous or not. Consider the more physically reasonable model of heat conduction where the conductivity k depends on the temperature u:

$$\frac{\partial}{\partial x}\left[k(u)\frac{\partial u}{\partial x}\right] = c_v\,\frac{\partial u}{\partial t}.$$

Suppose that $k(u) = k_0 u$ for constants k_0 and c_v. Rewrite the equation by differentiating and letting $\alpha = c_v/k_0$. If two solutions, $u_1(x,\, t)$ and $u_2(x,\, t)$, are known is their sum a solution? Why?