M.S. Comprehensive Examination
Spring 2002

Instructions:

1. If you think that there is a mistake ask the proctor. If the proctor's explanation is not satisfactory, interpret the problem as you see fit, but not in such a way that the answer is trivial.
2. From each part solve 3 of 5 problems.
3. If you solve more that three problems from a part, indicate the problems that you wish to have graded.

Part A

1. Consider the system

$$\begin{aligned} \dot{x} &= \sin(t)y \\ \dot{y} &= -\cos(t)x \end{aligned}$$

Suppose a solution $u(t) = (x(t), y(t))$ has initial values $u(0) = u_0 = (0, 1)$. Use the Fundamental Inequality to show that for $0 \leq t \leq \pi$

$$||u(t) - u_0|| \leq \frac{1}{2}(1 - \sqrt{2}\cos(t - \frac{\pi}{4}))$$

2. Consider the system $\dot{x} = (1-x^2)a(t)$ for a continuous function $a(t)$. What condition on $a(t)$ implies that the solution $x(t) = 1$ is Lyapunov stable. What condition implies that $x(t) = 1$ is asymptotically stable. Find an $a(t)$ so that $x(t) = 1$ is uniformly asymptotically stable.

3. Consider the system

$$\begin{aligned} \dot{x} &= -1 - x^2 + z^2 \\ \dot{y} &= 1 - y^2 + x^2 \\ \dot{z} &= 3 - 4xy - 3z^2. \end{aligned}$$

Find the stationary points and determine which are asymptotically stable.

4. Show that the unit circle is a limit cycle for the following equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} (1-r^2)x & -r^2 y \\ r^2 x & +(1-r^2)y \end{bmatrix}.$$

where $r = \sqrt{x^2+y^2}$.

5. Find the fundamental solution of the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 3x & & +4z \\ 2x & & +3z \\ -2x & +y & -2z \end{bmatrix}.$$

Part B

1.

(a) Solve the Cauchy problem $u_x + \frac{1}{u}u_y = u^2$ with the initial conditions $u(x,1) = 1$

(b) What condition on the initial data guarantees the existence of a solution in a neighborhood of the initial curve. Is this condition satisfied in this problem?

2. Find the canonical form and the general solution of the equation

$$2xu_{xx} + 2(1+xy)u_{xy} + 2yu_{yy} + \frac{2(1-x)}{1-xy}u_x + \frac{2(1-y)}{1-xy}u_y = 0.$$

3.

(a) Find the solutions to the Dirichlet problem $\triangle u + 5u = 0$ and $u|_{\partial R} = 0$ where $R = \{(x,y)|0 < x < \pi, 0 < y < \pi\}$

(b) What property of solutions to the Laplace equation on R is not shared with solutions to this equation. What feature of this equations causes this property to fail.

4. Let A be a 2×2 matrix that has the real Jordan form $\begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}$. For $u(x,y) = \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix}$ describe the general solution to the Cauchy-Kowalewski system

$$\frac{\partial}{\partial x} u(x,y) = A \frac{\partial}{\partial y} u(x,y).$$

5. Describe the symbol of the minimal surface equation

$$(1 + {u_y}^2) u_{xx} - 2 u_x u_y u_{xy} + (1 + {u_x}^2) u_{yy} = 0$$

and show that it is elliptic.