

Fall 1993

MA Comprehensive Exam in Algebra

Directions: Do all the problems. Write clearly and concisely. Be clear in your writing about what results you are using.

The examiners have made every effort to make sure that the problems are correct. However, if you are convinced that a problem contains a misprint, explain why you think so, formulate a correct and non-trivial version of the problem, and do that instead.

Q = the rational numbers

\mathbf{R} = the real numbers

C = the complex numbers

1. (a) If G is a group which contains only a finite number of subgroups, show that G is finite.
(b) Describe all groups which contain no proper subgroups.
(c) Describe all groups G which contain exactly one proper non-trivial subgroup.
2. (a) Show that the alternating group A_4 contains a normal subgroup of order 4.
(b) Prove or disprove: There is a homomorphism from A_4 onto $Z/(3)$ (where $Z/(3)$ is the group of integers mod 3).
(c) Prove or disprove: There is a homomorphism from A_4 onto $Z/(4)$.
(d) Prove or disprove: There is a homomorphism from A_4 onto S_3 , the symmetric group on three letters.
3. (a) If R is an integral domain and I and J are non-zero ideals of R , prove that $I \cap J \neq 0$.
(b) If $R = Q[x]$, find all proper ideals I and J such that $I \cap J = (x^2 - x - 1)$ (the principal ideal generated by $x^2 - x - 1$). Is your answer the same if $R = \mathbf{R}[x]$?
(c) If $R = Q[x]$, prove or disprove the statement that $R = (x^4 - x^2) + I$ for some proper ideal I .
4. Prove that a finite integral domain is a field.

5. Let V be a finite dimensional vector space over C and $T : V \rightarrow V$ be a linear transformation satisfying the equation $T^4 = I$.
- (a) Prove that T can be represented by a diagonal matrix.
 - (b) Give an example to show that if V is a finite dimensional vector space over \mathbf{R} , and T is as above, then T need not be diagonalizable.
6. Suppose $f : V \rightarrow W$ and $g : W \rightarrow X$ are linear transformations, where V , W and X are vector spaces of dimensions n , k and m respectively. Suppose that f and g have ranks r_f and r_g respectively. What are the possible ranks for the composite gf ? Justify your answer.