MS COMPREHENSIVE EXAM
DIFFERENTIAL EQUATIONS
SPRING 2011

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This exam has two parts, (A) ordinary differential equations and (B) partial differential equations. Do any three of the four problems from each part. Clearly indicate which three are to be graded and show the details of your work.

Part A: Ordinary Differential Equations

1. Consider the initial value problem

$$u'' - (1-u^2)u' + u = 0,\ u(0) = 1,\ u'(0) = 1.$$

   (a) Convert the initial value problem given above to a first order system of ordinary differential equations with initial conditions.

   (b) Apply one step of Euler's numerical method for vectors with stepsize $h = .01$ to the first order system with initial conditions found in part (a) above.

   (c) Use the results of part (b) above to estimate the values of $u(.01)$ and $u'(.01)$ in the original initial value problem.

2. (a) Find all critical points of $\begin{cases} x' = x + y^2 \\ y' = x^2 - y \end{cases}$.

   (b) Classify the critical point $x = -1$, $y = 1$ as to the type and stability. Refer to the attached table. Provide a phase plane portrait showing the direction of increasing t.

3. Consider the second order linear homogeneous ordinary differential equation

$$(1-t^2)y'' - 2ty' + 12y = 0.$$

Assume a solution of the form

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \cdots$$

to find two linearly independent solutions.

4. Let y_1 and y_2 be two differentiable functions defined on (a, b). Suppose that the Wronskian

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

is nonzero where t_0 is in (a, b).

Show that y_1 and y_2 are linearly independent on interval (a, b) by showing that

$$c_1 y_1(t) + c_2 y_2(t) = 0 \text{ for all } t \text{ in } (a, b)$$

implies $c_1 = c_2 = 0$.

Part B: Partial Differential Equations

You may work any three problems completely.

1. Suppose a metal rod, not laterally insulated, has an initial temperature of 20°C but immediately thereafter has one end fixed at 50°C. The rest of the rod is immersed in a liquid solution of temperature 30°C. Give the IVBP that describes this problem.

2. Prove that the problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0, && 0 < x < L,\ t > 0 \\ u(0,t) &= f_1(t) && 0 \le x \le L \\ u_x(L,t) &= f_2(t) && 0 \le x \le L \\ u(x,0) &= f_3(x) && \\ u_t(x,0) &= f_4(x) && \end{aligned}$$

   has at most one solution.

3. Solve by separating variables:

$$\begin{aligned} u_{tt} - u_{xx} - u(x,t) &= 0 \qquad 0 < x < 1,\ t > 0 \\ u_t(x,0) &= 0 \\ u(0,t) = u(1,t) &= 0 \end{aligned}$$

4. Solve the two equations separately

$$u_{xy} = 1 \text{ and } u_{yx} = 1.$$

   Make a conclusion as to the mixed derivatives.