Probability and Statistical Theory

MS Comprehensive Examination

December 13, 2006

Instructions:

Please answer all ~~three~~ 4 questions.

Record your answers in your blue books.

Show all of your computations.
Prove all of your assertions or quote the appropriate theorems.
Books, notes, and calculators *may be used.*

You have three hours.

1. (25). Let Z be a continuous random variable with pdf
$f(z) = \beta e^{-\beta z}, 0 < z < \infty$
where $\beta > 0$. Define $X = \alpha e^Z$ with $\alpha > 0$.
1). Show that X has density $f(x) = \beta\alpha^\beta x^{-(\beta+1)}$.

2). Find the expected value of X assuming $\beta > 1$.

3). Derive the conditional distribution of X given that $X < x_0$ with $x_0 > \alpha$.

4). Let $X_1, \cdots X_n$ be iid random variables with the density given in 1) and let $Y_{(1:n)} = \min(X_1, \cdots X_n)$. Show that the limiting distribution of $Y_{(1:n)}$ as n tends to infinity is a degenerate distribution.

2. (25) The joint distribution of X and Y is given by the following density
$f(x, y) = 60x^2y, 0 < x < 1, 0 < y < 1, 0 < x + y < 1.$

1). Are X and Y independent random variables? Explain.

2). Find the marginal distribution of X.

3). Define $U = \frac{X}{X+Y}$ and $V = X + Y$. Find the joint pdf of U and V.

4). Find $E(X|X + Y = k)$.

4. (30) Let $X_1, \ldots, X_{16}$ denote a random sample of size n=16 from the distribution over the interval [0,1] with density function $f(x) = cx^{\alpha}$ with parameter $\alpha > -1$; c is a fixed constant to be determined.

a. Show that $c = \alpha + 1$.

b. Show that for this distribution the mean $\mu = (\alpha+1)/(\alpha+2)$ and the variance $\sigma^2 = (\alpha+1)/(\alpha+3)(\alpha+2)^2$.

c. Show that the method of moments estimator of α is $\hat{\alpha}_{mm} = (2\bar{x}-1)/(1-\bar{x})$. Note how difficult it might be to calculate moments for this estimator. So, ... (see part d)....

d. Show that the first order Taylor series expansion of this method of moments estimator about $\bar{x} = ½$ is $\hat{\alpha}_{mm,ts(1/2)} = 4\bar{x} - 2$.

e. We will ultimately be interested in testing the null hypothesis that $\alpha = 0$. For $\alpha = 0$, use this Taylor series approximation along with the central limit theorem and the moments in part b to identify an approximate distribution for $\bar{x}$.

f. Note from part e that if $\alpha = 0$, then $\bar{x}$ is near ½, making the Taylor series expansion in part d valid. For α near 0, show that the bias of $\hat{\alpha}_{mm,ts(1/2)}$ is $-\alpha^2/(\alpha+2)$.

g. Show that the product P of the 16 observations is a sufficient statistic for α.

h. Show that the maximum likelihood estimator for α is $-(1+1/A)$ where $A = \ln(P)/n$, which is the sample average of the $\ln(X_i)$.

i. Find the likelihood ratio test statistic for testing H_0: $\alpha = 0$ versus the alternative H_A: $\alpha \neq 0$. Is this a function of the sufficient statistic S?

j. Using the typical asymptotic approximation, give the result of this test of the dataset below. Use level of significance $\alpha = .10$. What is your conclusion?

Data:

x	ln(x)
0.385930	-0.95210
0.889914	-0.11663
0.995331	-0.00468
0.972367	-0.02802
0.107445	-2.23078
0.652923	-0.42630
0.702398	-0.35326
0.206126	-1.57927
0.201337	-1.60277
0.194381	-1.63793
0.396993	-0.92384
0.749426	-0.28845
0.125019	-2.07929
0.435637	-0.83095
0.853433	-0.15849
0.022832	-3.77957

Statistics:

Variable	N	Mean	StDev	Minimum	Q1	Median	Q3	Maximum
x	16	0.4932	0.3357	0.0228	0.1961	0.4163	0.8274	0.9953
ln(x)	16	-1.062	1.038	-3.780	-1.629	-0.877	-0.191	-0.00468

#. (20) Consider the following experiment: In a sequence of independent trials, roll a fair, four-sided, die (outcomes in {1,2,3,4} all equally likely on each trial) and simultaneously toss a fair coin. Perform these trials only until the first time a "T" is tossed on the coin. So we roll until the coin comes up tails. Let S denote the sum of all of the results on the die, up to and including that last roll. Our aim is to find the expectation and standard deviation of S. The problem consists of some steps along the way, as detailed below.

a. Denote by Z the number of trials performed. Name and write down the distribution of Z.
b. Derive E(Z) and Var(Z). Use any method you like, but show the derivation.
c. Let X_i denote the number rolled on the die on trial 'i'. Note that $i \leq Z$. Write down the conditional distribution, expectation, and variance of X_i given $i \leq Z$.
d. Use the formula $E(S) = E(E(S|Z))$ to find E(S).
e. Use the formula $Var(S) = Var(E(S|Z)) + E(Var(S|Z))$ to finish the problem by finding the standard deviation of S.