Topology I

Mao-Pei Tsui

University of Toledo

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Closed Sets

Definition: *A subset F of a topological space X is said to be closed if its complement $X \backslash F$ is open.*

Remark; (a) From the definition of topological spaces, the following properties follow immediately:

(i) X and $\emptyset$ are closed,

(ii) Finite unions of closed sets are closed,

(iii) Arbitrary intersections of closed sets are closed.

(b) A topology on a set X can be defined by describing the collection of closed sets, as long as they satisfy these three properties; the open sets are then just those sets whose complements are closed.

We can also determine the continuity of a function using closed sets.

Lemma 2.7. *A map between topological spaces is continuous if and only if the inverse image of every closed set is closed.*

Given any set $A \subset X$, we define the **closure, interior, exterior, boundary** of A as follows.

Definition: *The closure of A in X, denoted by $\overline{A}$ is the set* $\overline{A} = \cap\{B \subset X : A \subset \ B \text{ and } B \text{ is closed in } X\}$.

The interior of A, written $IntA$, is $Int\ A = \cup\{C : C \subset A \text{ and} C \text{ is open in } X\}$.

A point $q \in X$ is a limit point of A if every neighborhood of q contains a point of A other than q (which might or might not itself be in A).

Remark: It is obvious from the properties of open and closed sets that $\overline{A}$ is closed and *Int* A is open.

We also have the following two facts:

$\overline{A}$ is the smallest closed set containing A

Int A is the largest open set contained in A.

Definition: *The exterior of* A *,written* $ExtA$ *as* $ExtA = X \backslash \overline{A}$

and the boundary of A*, written* ∂A*, as* $\partial A = X \backslash (IntA \cup ExtA)$*.*

Remark: $X = IntA \cup ExtA \cup \overline{A}$ (b/c $\partial A = X \backslash (IntA \cup ExtA)$).

For many purposes, it is useful to have alternative characterizations of open and closed sets, and of the interior, exterior, closure, and boundary of a given set. The following lemma gives such characterizations.

Lemma 2.8. *Let X be a topological space and $A \subset X$ any subset.*

(a) A point $q \in \text{int } A$ iff q has a neighborhood contained in A.

(b) A point $q \in \text{ext } A$ iff q has a neighborhood contained in $X \backslash A$.

(c) A point $q \in \partial A$ iff every neighborhood of q contains both a point of A and a point of $X \backslash A$.

(d) Int A and Ext A are open in X , while $\overline{A}$ is closed in X.

(e) A is open iff $A = IntA$,

(f) A is closed iff it contains all its boundary points, which is true if and only if $A = IntA \cup \overline{A} = \overline{A}$.

(g) $\overline{A} = A \cup \partial A = Int\ A \cup \partial A$.

(h) A set A in a topological space is closed iff it contains all of its limit points.

Definition: *A subset A of a topological space X is said to be dense in X if* $\overline{A} = X$.

Example. Show that a subset $A \subset X$ is dense if and only if every nonempty open set in X contains a point of A.

Proof: This follows from the fact that a point $q \in \overline{A}$ iff every neighborhood of q contains both a point of A and a point of $X \backslash A$.

Example. The set of all points with rational coordinates is a countable dense subset in R^n.

To define a topology on a given set, it is often convenient to single out some "special" open sets and use them to define the rest of the open sets. For example, the metric topology is defined by first defining balls and then declaring a set to be open if it contains a ball around each of its points. This idea can be generalized easily to arbitrary topological spaces, as in the next definition.

Definition: *Suppose X is any set. A basis in X is a collection $\mathcal{B}$ of subsets of X satisfying the following conditions:*

(i) Every element of X is in some element of $\mathcal{B}$; in other words, $X = \cup_{B \in \mathcal{B}} B$.

(ii) If B_1, $B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Proposition 2.9. *Let $\mathcal{B}$ be a basis in a set X, and let $\mathcal{T}$ be the collection of all unions of elements of $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X.*

This topology T is called the topology generated by the basis $\mathcal{B}$.