Finding the limit of a function

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- $\lim_{x\to 1} \frac{x^2-3x+2}{x-1}$
  Plugging 1 to the expression, we get $\frac{1-3+2}{1-1} = \frac{3-3}{1-1} = \frac{0}{0}$.

  This means that we can not determine the limit just by plugging in the number 1.

  In this case, we can factor $x^2 - 3x + 2 = (x-1)(x-2)$.
  to get $\frac{x^2-3x+2}{x-1} = \frac{(x-1)(x-2)}{x-1} = x - 2$.

  So $\lim_{x\to 1} \frac{x^2-3x+2}{x-1} = \lim_{x\to 1} x - 2 = 1 - 2 = -1$.

- $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1}$

  Plugging 1 to the expression, we get $\frac{\sqrt{1+3}-2}{1-1} = \frac{2-2}{1-1} = \frac{0}{0}$. This means that we can not determine the limit just by plugging in the number 1.

  In this case, we need to rationalize the expression by multiplying $\sqrt{x+3}+2$ to the top and the bottom to get

  $$\frac{\sqrt{x+3}-2}{x-1} = \frac{\sqrt{x+3}-2}{x-1} \cdot \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \frac{(\sqrt{x+3})^2-2^2}{(x-1)(\sqrt{x+3}+2)} = \frac{(x+3)-4}{(x-1)(\sqrt{x+3}+2)}$$

  $$= \frac{x-1}{(x-1)(\sqrt{x+3}+2)} = \frac{\overbrace{(x-1)}}{\underbrace{(x-1)}(\sqrt{x+3}+2)} = \frac{1}{(\sqrt{x+3}+2)}.$$

  So $\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x\to 1} \frac{1}{(\sqrt{x+3}+2)} = \frac{1}{\sqrt{1+3}+2} = \frac{1}{4}$.

- $\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1}$

  Plugging 1 to the expression, we get $\frac{\sqrt{1+3}-2}{1-1} = \frac{2-2}{1-1} = \frac{0}{0}$.
  This means that we can not determine the limit just by plugging in the number 1.
  In this case, we need to rationalize the expression by multiplying $\sqrt{x^2+3}+2$ to the top and the bottom to get $\frac{\sqrt{x^2+3}-2}{x-1} = \frac{\sqrt{x^2+3}-2}{x-1} \cdot \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} = \frac{\left(\sqrt{x^2+3}\right)^2-2^2}{(x-1)(\sqrt{x^2+3}+2)} = \frac{(x^2+3)-4}{(x-1)(\sqrt{x^2+3}+2)}$

  $$= \frac{x^2-1}{(x-1)(\sqrt{x^2+3}+2)} = \frac{\underbrace{(x-1)}(x+1)}{\underbrace{(x-1)}(\sqrt{x^2+3}+2)} = \frac{(x+1)}{(\sqrt{x^2+3}+2)}.$$

  So $\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x-1} = \lim_{x\to 1} \frac{(x+1)}{(\sqrt{x^2+3}+2)} = \frac{1+1}{\sqrt{1+3}+2} = \frac{2}{4} = \frac{1}{2}$.

- $\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$

  Plugging 1 to the expression, we get $\frac{\frac{1}{1+1}-\frac{1}{2}}{1-1} = \frac{0}{0}$.

  This means that we can not determine the limit just by plugging in the number 1.
  In this case, we need to simplify the expression
  to get $\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = \frac{\frac{2-(x+1)}{2(x+1)}}{x-1} = \frac{\frac{2-x-1}{2(x+1)}}{x-1} = \frac{\overbrace{1-x}}{2(x+1)\underbrace{(x-1)}}$

  $= \frac{-1}{2(x+1)}$.
  So $\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{-1}{2(x+1)} = -\frac{1}{4}$.

Review of simple algebra:

$a - (b + c) = a - b - c$

$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a \cdot d}{b \cdot c}$

$\frac{\frac{a}{b}}{c} = \frac{a}{b \cdot c}$.

Example. $f(x) = -x^2 - 3x + 3$. Find $\frac{f(-2+h)-f(-2)}{h}$.
Solution: First, notice that $f(\square) = -\square^2 - 3\square + 3$. So
$f(-2 + h) = -(-2 + h)^2 - 3(-2 + h) + 3 =$
$-(4-4h+h^2)+6-3h+3 = -4+4h-h^2+6-3h+3 = 5+h-h^2$
and $f(-2) = -(-2)^2 - 3(-2) + 3 = -4 + 6 + 3 = 5$.

Hence $\frac{f(-2+h)-f(-2)}{h} = \frac{5+h-h^2-5}{h} = \frac{h-h^2}{h} = \frac{h(1-h)}{h} = 1 - h$.

- $\lim_{x\to 1} \frac{\frac{1}{x^2+1}-\frac{1}{2}}{x-1}$

  Plugging 1 to the expression, we get $\frac{\frac{1}{1+1}-\frac{1}{2}}{1-1} = \frac{0}{0}$.

  This means that we can not determine the limit just by plugging in the number 1.
  In this case, we need to simplify the expression
  to get $\frac{\frac{1}{x^2+1}-\frac{1}{2}}{x-1} = \frac{\frac{2-(x^2+1)}{2(x^2+1)}}{x-1} = \frac{\frac{2-x^2-1}{2(x^2+1)}}{x-1} = \frac{\underbrace{1-x^2}}{2(x^2+1)\underbrace{(x-1)}}$

  $= \frac{\underbrace{(1-x)}(1+x)}{2(x^2+1)\underbrace{(x-1)}} = \frac{-1\cdot(1+x)}{2(x^2+1)}$.

  So $\lim_{x\to 1} \frac{\frac{1}{x^2+1}-\frac{1}{2}}{x-1} = \lim_{x\to 1} \frac{-(1+x)}{2(x^2+1)} = -\frac{2}{4} = -\frac{1}{2}$.