Differentiation rules

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Today's topics:

- Derivation of product rule $(uv)' = u'v + uv'$

- Practice differentiation rules

Product rule

Goal: To prove the product rule
$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$.

- Let $f(x) = u(x)v(x)$.

  We want to find the derivative of $f(x)$.

  From the definition of the derivative, we have
  $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$.

  Since $f(x) = u(x)v(x)$, we have $f(x+h) = u(x+h)v(x+h)$

  and $f(x+h) - f(x) = u(x+h)v(x+h) - u(x)v(x)$.

- Now we express the change of f in terms of the changes of u and v.

$$f(x+h) - f(x)$$

$$= u(x+h)v(x+h) - u(x)v(x)$$

$$= u(x+h)v(x+h) \underbrace{-u(x)v(x+h) + u(x)v(x+h)}_{=0} - u(x)v(x)$$

(we add the term $-u(x)v(x+h) + u(x)v(x+h)$ which is 0.)

$$= \Big(u(x+h) - u(x)\Big)v(x+h) + u(x)\Big(v(x+h) - v(x)\Big)$$

(We group the first two terms and the last two terms together.)

- Now divide by h and let $h \to 0$:

$$lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\Big(u(x+h)-u(x)\Big)v(x+h)+u(x)\Big(v(x+h)-v(x)\Big)v(x)}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}v(x+h)+u(x)\frac{v(x+h)-v(x)}{h}$$

- (use the limit properties)

$$= \left(\lim_{h\to 0} \frac{u(x+h)-u(x)}{h}\right)\left(\lim_{h\to 0} v(x+h)\right)$$

$$+u(x)\left(\lim_{h\to 0} \frac{v(x+h)-v(x)}{h}\right)$$

$$= u'(x)v(x) + u(x)v'(x),$$

as claimed. In this last step we have used that

$$\lim_{h\to 0} \frac{u(x+h)-u(x)}{h} = u'(x) \quad \text{and} \quad \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} = v'(x)$$

and also that $\lim_{h\to 0} v(x+h) = v(x)$.

This last limit follows from the fact that v is continuous, which in turn follows from the fact that v is differentiable. The following is the proof.

$$
\begin{aligned}
&\lim_{h\to 0} v(x+h) - v(x) \\
=&\lim_{h\to 0} h\Big(\frac{v(x+h)-v(x)}{h}\Big) \\
=&\lim_{h\to 0} h \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} \\
=&0\cdot f'(x) \\
=&0
\end{aligned}
$$

Thus $\lim_{h\to 0} v(x+h) - v(x) = 0$ which is the same as $\lim_{h\to 0} v(x+h) = v(x)$.

Constant rule:	$c' = 0$	$\frac{dc}{dx} = 0$
Power rule rule:	$(cx^a)' = c \cdot a \cdot x^{a-1}$	$\frac{d}{dx}(cx^a) = c \cdot a \cdot x^{a-1}$
Sum rule:	$(u \pm v)' = u' \pm v'$	$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$
Product rule:	$(u \cdot v)' = u' \cdot v + u \cdot v'$	$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$
Quotient rule:	$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$	$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Table: The differentiation rules

Practice differentiation rules

- Open the file by clicking `Z:/Math/Math1850/M4C13/MapletsForCalculus/MapletsForCalculus/files/maplets/DerivativeDrill.maplet`
- We will use this Maplet to practice differentiation rules.
- First, click on "Functions" to choose only Polynomials.
- Now click on "Rules" to choose "Product Rules".
- Now click on "New Derivative function".
- When you type your answer, you don't have to simplify your answer. For example, if the problem is $\frac{d}{dt}((-2t^3+2t+1)(3t^2-4t+1))$.
  You should type
  $(-6*t^\wedge 2+2)*(3*t^\wedge 2-4*t+1)+(-2*t^\wedge 3+2*t+1)*(6*t-4)$
  as your answer.
- Now do 10 problems.
- I or TA will give you a quiz when you are done.

Picture of the Product Rule. If u and v are quantities which depend on x, and if increasing x by Δx causes u and v to change by Δu and Δv, then the product of u and v will change by

$$\Delta(uv) = (u+\Delta u)(v+\Delta v) - uv = u\Delta v + v\Delta u + \Delta u \Delta v. \quad (0.0.1)$$

If u and v are differentiable functions of x, then the changes Δu and Δv will be of the same order of magnitude as Δx, and thus one expects $\Delta u \Delta v$ to be much smaller. One therefore ignores the last term in (0.0.1), and thus arrives at

$$\Delta(uv) = u\Delta v + v\Delta u.$$

Leibniz would now divide by Δx and replace Δ's by d's to get the product rule:

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x}.$$

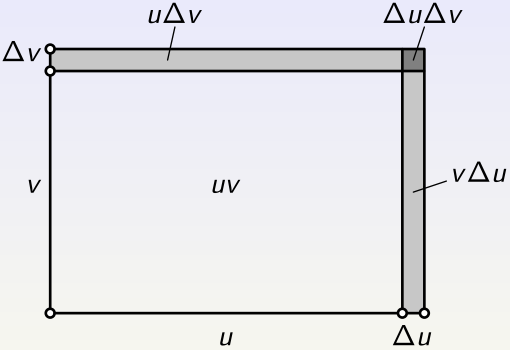

Figure: The Product Rule. *How much does the area of a rectangle change if its sides u and v are increased by Δu and Δv?* Most of the increase is accounted for by the two thin rectangles whose areas are $u\Delta v$ and $v\Delta u$. So the increase in area is approximately $u\Delta v + v\Delta u$, which explains why the product rule says $(uv)' = uv' + vu'$.