Solution to Review Problems for Midterm #1

Midterm I: Wednesday, September 22 in class
Topics: 1.1, 1.3 and 2.1-2.6 (except 2.3)
Office hours before the exam: Monday 11-1 and 4-6 p.m., Tuesday 1-2 pm and 4-6 pm at UH 2080B)

Topics1 Find the equation of the tangent line to $y = f(x)$ at $x = a$,
1. We need to find the slope of the tangent line by finding the limit $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
2. Use the point slope formula to find the equation of the tangent line thru $(a, f(a))$ with slope m:
$y - f(a) = m(x - a)$.

1. Let $f(x) = -2x^2 + 3x + 1$. Find an equation of the tangent line to the curve at $P(1, f(1))$.
Solution: To find the slope of the tangent line at $x = 1$, we need to find the limit $m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$. Since $f(\square) = -2\square^2 + 3\square + 1$, we have $f(1+h) = -2(1+h)^2 + \underbrace{3(1+h)+1} = -2(1+2h+h^2) + \underbrace{3+3h+1} =$
$-2-4h-2h^2 + \underbrace{4+3h} = 2-h-2h^2$, $f(1) = -2+3+1 = 2$ and
$f(1+h) - f(1) = 2-h-2h^2-2 = -h-2h^2 = h(-1-2h)$. So $m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{h(-1-2h)}{h} = \lim_{h\to 0} -1-2h = -1$. Now we have the point $P = (1, f(1)) = (1, 2)$ and the slope of the tangent line $m = -1$. By the point slope formula, we have the equation of the tangent line $y - 2 = -1(x-1) = -x+1$ and $y = -x+1+2 = -x+3$. Hence the equation of the tangent line to the curve at $P(1, f(1))$ is $y = -x+3$.

2. Let $f(x) = \frac{2}{x^2+1}$. Find an equation of the tangent line to the curve at $P(1, f(1))$.
Solution: To find the slope of the tangent line at $x = 1$, we need to find the limit $m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$. Since $f(\square) = \frac{2}{\square^2+1}$, we have $f(1+h) = \frac{2}{(1+h)^2+1} = \frac{2}{(1+2h+h^2)+1} = \frac{2}{2+2h+h^2}$, $f(1) = \frac{2}{1^2+1} = \frac{2}{2} = 1$ and
$f(1+h) - f(1) = \frac{2}{2+2h+h^2} - 1 = \frac{2}{2+2h+h^2} - \frac{\mathbf{2+2h+h^2}}{2+2h+h^2} = \frac{2-(\mathbf{2+2h+h^2})}{2+2h+h^2} = \frac{2-2-2h-h^2}{2+2h+h^2} =$
$\frac{-2h-h^2}{2+2h+h^2} = \frac{h(-2-h)}{2+2h+h^2}$. So $m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{h(-2-h)}{2+2h+h^2}}{h} = \lim_{h\to 0} \frac{h(-2-h)}{(2+2h+h^2)h} =$
$\lim_{h\to 0} \frac{(-2-h)}{(2+2h+h^2)} = \frac{-2-0}{2+0+0} = -1$. Now we have the point $P = (1, f(1)) = (1, 1)$ and the slope of the tangent line $m = -1$. By the point slope formula, we have the equation of the tangent line $y-1 = -1(x-1) = -x+1$ and $y = -x+1+1 = -x+2$. Hence the equation of the tangent line to the curve at $P(1, f(1))$ is $y = -x+2$.

3. Let $f(x) = \frac{2}{\sqrt{x^2+3}}$. Find an equation of the tangent line to the curve at $P(1, f(1))$.

Solution: To find the slope of the tangent line at $x = 1$, we need to find the limit $m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$. Since $f(\Box) = \frac{2}{\sqrt{\Box^2+3}}$, we have $f(1+h) = \frac{2}{\sqrt{(1+h)^2+3}} = \frac{2}{\sqrt{(1+2h+h^2)+3}} = \frac{2}{\sqrt{4+2h+h^2}}$, $f(1) = \frac{2}{\sqrt{1^2+3}} = \frac{2}{2} = 1$ and $f(1+h) - f(1) = \frac{2}{\sqrt{4+2h+h^2}} - 1 = \frac{2}{\sqrt{4+2h+h^2}} - \frac{\sqrt{4+2h+h^2}}{\sqrt{4+2h+h^2}} = \frac{2-\sqrt{4+2h+h^2}}{\sqrt{4+2h+h^2}}$. Now we rationalize the expression by multiplying $\frac{2+\sqrt{4+2h+h^2}}{2+\sqrt{4+2h+h^2}}$ to get

$f(1+h) - f(1) = \frac{2-\sqrt{4+2h+h^2}}{\sqrt{4+2h+h^2}} \frac{2+\sqrt{4+2h+h^2}}{2+\sqrt{4+2h+h^2}} = \frac{2^2-(4+2h+h^2)}{(\sqrt{4+2h+h^2})(2+\sqrt{4+2h+h^2})}$
$= \frac{4-4-2h-h^2)}{(\sqrt{4+2h+h^2})(2+\sqrt{4+2h+h^2})} = \frac{-2h-h^2)}{(\sqrt{4+2h+h^2})(2+\sqrt{4+2h+h^2})}$.

So $m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{-2h-h^2)}{(\sqrt{4+2h+h^2})(2+\sqrt{4+2h+h^2})}}{h}$
$= \lim_{h\to 0} \frac{h(-2-h))}{(\sqrt{4+2h+h^2})(2+\sqrt{4+2h+h^2})h}$ (factoring out h from the top and the bottom) $= \lim_{h\to 0} \frac{(-2-h)}{(\sqrt{4+2h+h^2})(2+\sqrt{4+2h+h^2})} = \frac{-2-0}{\sqrt{4}\cdot(2+\sqrt{4})} = \frac{-2}{2\cdot 4} = -\frac{1}{4}$. Now we have the point $P = (1, f(1)) = (1, 1)$ and the slope of the tangent line $m = -\frac{1}{4}$. By the point slope formula, we have the equation of the tangent line $y - 1 = -\frac{1}{4}(x-1) = -\frac{1}{4}x + \frac{1}{4}$ and $y = -\frac{1}{4}x + \frac{1}{4} + 1 = -\frac{x}{4} + \frac{5}{4}$. Hence the equation of the tangent line to the curve at $P(1, f(1))$ is $y = -\frac{x}{4} + \frac{5}{4}$.

Topic2 Left continuity: To determine if a function is left continuous at $x = a$, we need to find $lim_{x\to a^-} f(x)$ and $f(a)$. If $lim_{x\to a^-} f(x) \neq f(a)$ then f is not left continuous. If $lim_{x\to a^-} f(x) = f(a)$ then f is left continuous. Right

continuity at $x = a$: To determine if a function is right continuous at $x = a$, we need to find $lim_{x\to a^+} f(x)$ and $f(a)$. If $lim_{x\to a^+} f(x) \neq f(a)$ then f is not left continuous. If $lim_{x\to a^+} f(x) = f(a)$ then f is left continuous. Continuity at $x = a$:To determine if a function is t con-

tinuous at $x = a$, we need to find $lim_{x\to a^-} f(x)$, $lim_{x\to a^+} f(x)$ and $f(a)$. If $lim_{x\to a^-} f(x) \neq lim_{x\to a^+} f(x)$ then f is not continuous. (either jump or infinite discontinuity). If $lim_{x\to a^-} f(x) = lim_{x\to a^+} f(x) \neq f(a)$ then f is not continuous. (This is called the removable discontinuity.) If $lim_{x\to a^-} f(x) = lim_{x\to a^+} f(x) = f(a)$ then f is continuous.

4. A piecewise defined function is given by

$$f(x) = \begin{cases} -x-1, & x < -1 \\ x^2 - 1, & -1 \le x < 2 \\ x+2, & 2 \le x \end{cases}$$

Determine if f is left continuous, right continuous or continuous at $x = -1$ or $x = 2$.

Solution: At $x = -1$, we have $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -x - 1 = 1 - 1 = 0$, $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x^2 - 1 = (-1)^2 - 1 = 0$ and $f(1) = 1^2 - 1 = 0$. So $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$. Hence f is left continuous, right continuous and continuous at $x = -1$.

At $x = 2$, we have $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 - 1 = 4 - 1 = 3$, $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x + 2 = 2 + 2 = 4$ and $f(2) = 2 + 2 = 4$. So $\lim_{x \to 2^-} f(x) \neq f(2)$ and f is not left continuous at $x = 2$. Now $\lim_{x \to 2^+} f(x) = f(2)$ and f is right continuous $x = 2$.

But $\lim_{x \to 2^-} f(x) = 3 \neq 4 = \lim_{x \to 2^+} f(x)$ and f has a jump discontinuity at $x = 2$.

5. Classify the discontinuity of the following functions (removable, infinite, jump or oscillating discontinuity). Redefine the value of the function if it's removable.

(a) $f(x) = \frac{x-3}{x^2-4x+3}$

Solution: Note that $x^2 - 4x + 3 = (x-1)(x-3)$. The domain of f is $\{x | x \neq 1 \text{ and } x \neq 3\}$. So f is discontinuous at $x = 1$ and $x = 3$. At $x = 1$, $\lim_{x \to 1^-} \frac{x-3}{x^2-4x+3} = \lim_{x \to 1^-} \frac{x-3}{(x-1)(x-3)} = \lim_{x \to 1^-} \frac{1}{(x-1)} = \frac{1}{0^-} = -\infty$ and $\lim_{x \to 1^+} \frac{x-3}{x^2-4x+3} == \lim_{x \to 1^+} \frac{1}{(x-1)} = \frac{1}{0^+} = \infty$. So f has an infinite discontinuity at $x = 1$.

At $x = 3$, $\lim_{x \to 3^-} \frac{x-3}{x^2-4x+3} = \lim_{x \to 3^-} \frac{1}{(x-1)} = \frac{1}{2}$ and $\lim_{x \to 3^+} \frac{x-3}{x^2-4x+3} = \lim_{x \to 3^+} \frac{1}{(x-1)} = \frac{1}{2}$. So f has an removable discontinuity at $x = 3$. We can define $f(3) = \frac{1}{2}$ to make f continuous at $x = 3$.

(b)

$$f(x) = \begin{cases} \dfrac{x+1}{x^2-1}, & x < -1 \\ \dfrac{x+1}{8}, & -1 \le x \le 1 \\ \dfrac{\sqrt{x}-1}{x^2-1}, & 1 < x \end{cases}$$

Solution: The only points where f may not be continuous is $x = -1$ and $x = 1$.

At $x = -1$, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x+1}{x^2-1} = \lim_{x\to -1^-} \frac{x+1}{(x+1)(x-1)} = \lim_{x\to -1^-} \frac{1}{x-1} = -\frac{1}{2}$, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \frac{x+1}{8} = 0$ and $f(-1) = \frac{-1+1}{8} = 0$. Hence $\lim_{x\to -1^-} f(x) \neq f(-1)$ and f is not left continuous at $x = -1$ Now $\lim_{x\to -1^+} f(x) = f(-1)$ and f is right continuous at $x = -1$. Since $\lim_{x\to -1^-} f(x) = -\frac{1}{2} \neq 0 = \lim_{x\to -1^+} f(x)$ and f has a jump discontinuity at $x = -1$.

At $x = 1$, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+1}{8} = \frac{2}{8} = \frac{1}{4}$, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x^2-1} = \lim_{x\to 1^+} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x^2-1)(\sqrt{x}+1)}$
$= \lim_{x\to 1^+} \frac{x-1}{(x^2-1)(\sqrt{x}+1)} = \lim_{x\to 1^+} \frac{x-1}{(x-1)(x+1)(\sqrt{x}+1)} = \lim_{x\to 1^+} \frac{1}{(x+1)(\sqrt{x}+1)} = \frac{1}{4}$ and $f(1) = \frac{1+1}{8} = \frac{1}{4}$. Since $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(1)$ and f has is left continuous, right continuous and continuous at $x = 1$.

6. A piecewise defined function is given by

$$f(x) = \begin{cases} x - 1, & x < -1 \\ ax + b, & -1 \leq x < 1 \\ x^2, & 1 \leq x \end{cases}$$

(a) Find the graph of $y = f(x)$ over the interval $(-\infty, -1) \cup [1, \infty)$.

(b) Determine the value of a and b so that f is continuous everywhere. Also explain your answer geometrically.
Solution: The only possible discontinuity are at $x = -1$ and $x = 1$. We first compute $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x - 1 = -1 - 1 = -2$, $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax + b = -a + b$ and $f(-1) = -a + b$ To make f continuous at $x = -1$, we must have $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$. This gives $-a + b = -2$.
Next we compute $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} ax+b = a+b$, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1$ and $f(1) = 1$ To make f continuous at $x = 1$, we must have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$. This gives $a + b = 1$.
Now we have two equations $-a + b = -2$ and $a + b = 1$. From $-a + b = -2$, we get $b = -2 + a$. Plugging $b = -2 + a$ to $a + b = 1$, we get $a + (-2 + a) = 1$, $2a = 3$ and $a = \frac{3}{2}$. Use $b = -2 + a$ to get $b = -2 + \frac{3}{2} = -\frac{1}{2}$. Thus $a = \frac{3}{2}$ and $b = -\frac{1}{2}$ will make f continuous everywhere.

Topic 3: Vertical asymptote and horizontal asymptote. To find vertical asymptote, we first find its domain. Let a be a point not in the domain. We find $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^=} f(x)$. If $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^+} f(x) = \infty$ then $x = a$ is a vertical asymptote.

To find horizontal asymptote, we find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$. If $\lim_{x\to-\infty} f(x) = b$ then $y = b$ is a horizontal asymptote. If $\lim_{x\to\infty} f(x) = c$ then then $y = c$ is a horizontal asymptote.

$\frac{1}{0^+} = \frac{negative\ number}{0^-} = \infty$, $\frac{1}{0^-} = \frac{positive\ number}{0^-} = -\infty$, $0^+ \cdot 0^- = 0^-$, $0^- \cdot 0^- = 0^+$.

$\frac{1}{\infty} = \frac{1}{-\infty} = 0$, $\infty \cdot \infty = -\infty \cdot -\infty = \infty$, $\infty \cdot -\infty = -\infty$, $(-\infty)^{odd\ power} = -\infty$. $x^p = x^n x^{-n+p}$, $\lim_{x\to\infty} x^{positive\ number} = \infty$, $\lim_{x\to\infty} x^{negative\ number} = 0$, $\lim_{x\to-\infty} x^{positive\ and\ even\ integer} = \infty$, $\lim_{x\to-\infty} x^{positive\ and\ odd\ integer} = -\infty$.

Given a rational function $R(x)$. Suppose we rearrange the order of the power in the top and the bottom so it's in decreasing order, i.e $R(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$. For example, $R(x) = \frac{x - x^3 + 2}{x - 5x^3 + 6x^4} = \frac{-x^3 + x + 2}{6x^4 - 5x^3 + x}$.

$lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} = lim_{x\to\pm\infty} \frac{x^n(a_n + a_{n-1}x^{-1} + \cdots + a_1 x^{-n+1} + a_0 x^{-n})}{x^m(b_m + b_{m-1}x^{-1} + \cdots + b_1 x^{-m+1} + b_0 x^{-m})} = \lim_{x\to\infty} \frac{a_n}{b_m} x^{n-m}$.

7. Determine the following limits

(a) $\lim_{x\to 1} \frac{x-1}{x^2-3x+2}$

Solution:$\lim_{x\to 1} \frac{x-1}{x^2-3x+2} = \lim_{x\to 1} \frac{x-1}{(x-1)(x-2)} = \lim_{x\to 1} \frac{1}{x-2} = -1$.

(b) $\lim_{x\to 2^+} \frac{x-4}{x^2-5x+6}$, $\lim_{x\to 2^-} \frac{x-4}{x^2-5x+6}$, $\lim_{x\to 2} \frac{x-4}{x^2-5x+6}$

Solution: Plugging in the expression, we get $\frac{2}{0}$. So we need to factor the bottom to analyze its behavior. $\lim_{x\to 2^+} \frac{x-4}{x^2-5x+6} = \lim_{x\to 2^+} \frac{x-4}{(x-2)(x-3)} = \frac{-2}{0^+ \cdot -1} = \frac{-2}{0^-} = \infty$.

$\lim_{x\to 2^-} \frac{x-4}{x^2-5x+6} = \lim_{x\to 2^-} \frac{x-4}{(x-2)(x-3)} = \frac{-2}{0^- \cdot -1} = \frac{-2}{0^+} = -\infty$.

Now we know that $\lim_{x\to 2^+} \frac{x-4}{x^2-5x+6} \neq \lim_{x\to 2^-} \frac{x-4}{x^2-5x+6}$. So $\lim_{x\to 2} \frac{x-4}{x^2-5x+6}$ doesn't exist.

(c) $\lim_{x\to\infty} \frac{-2x^2+x^6+1}{x^3-5x+6}$, $\lim_{x\to-\infty} \frac{-2x^2+x^6+1}{x^3-5x+6}$

Solution: $\lim_{x\to\infty} \frac{-2x^2+x^6+1}{x^3-5x+6} = \lim_{x\to\infty} \frac{x^6(-2x^{-4}+1+x^{-6})}{x^3(1-5x^{-2}+6x^{-3})} = \lim_{x\to\infty} \frac{x^3(-2x^{-4}+1+x^{-6})}{(1-5x^{-2}+6x^{-3})} = \lim_{x\to\infty} x^3 = \infty$. Note that we have used the fact that $\lim_{x\to\infty} \frac{(-2x^{-4}+1+x^{-6})}{(1-5x^{-2}+6x^{-3})} = 1$.

Similarly, $\lim_{x\to-\infty} \frac{-2x^2+x^6+1}{x^3-5x+6} = \lim_{x\to-\infty} \frac{x^6(-2x^{-4}+1+x^{-6})}{x^3(1-5x^{-2}+6x^{-3})} = \lim_{x\to-\infty} \frac{x^3(-2x^{-4}+1+x^{-6})}{(1-5x^{-2}+6x^{-3})} = \lim_{x\to-\infty} x^3 = (-\infty)^3 = -\infty$.

(d) $\lim_{x\to\infty} \frac{-2x^6+x^2+1}{4x-5x+6x^6}$, $\lim_{x\to-\infty} \frac{-2x^6+x^2+1}{4x-5x+6x^6}$
Solution: $\lim_{x\to\infty} \frac{-2x^6+x^2+1}{4x-5x+6x^6} = \lim_{x\to\infty} \frac{x^6(-2+x^{-4}+x^{-6})}{x^6(4x^{-3}-5x^{-5}+6)}$ (factoring out x^6 from the top and the bottom) $= \lim_{x\to\infty} \frac{(-2+x^{-4}+x^{-6})}{(4x^{-3}-5x^{-5}+6)} = \frac{-2}{6} = \frac{-1}{3}$.
Similarly, $\lim_{x\to-\infty} \frac{-2x^6+x^2+1}{4x-5x+6x^6} = \lim_{x\to-\infty} \frac{x^6(-2x^{-4}+1+x^{-6})}{x^6(4x^{-3}-5x^{-5}+6)} = \lim_{x\to-\infty} \frac{(-2x^{-4}+1+x^{-6})}{(4x^{-3}-5x^{-5}+6)} = \frac{1}{6}$.

(e) $\lim_{x\to\infty} \frac{-2x^6+x^2+1}{4x^3-5x^8+6}$, $\lim_{x\to-\infty} \frac{-2x^6+x^2+1}{4x^8-5x+6}$
Solution: $\lim_{x\to\infty} \frac{-2x^6+x^2+1}{4x^3-5x^8+6} = \lim_{x\to\infty} \frac{x^6(-2+x^{-4}+x^{-6})}{x^8(4x^{-5}-5+6x^{-8})} = \lim_{x\to\infty} \frac{(-2+x^{-4}+x^{-6})}{x^2(4x^{-5}-5+6x^{-8})} = \frac{-2}{\infty\cdot(-5)} = 0$.
Similarly, $\lim_{x\to-\infty} \frac{-2x^6+x^2+1}{4x^3-5x^8+6} = \lim_{x\to-\infty} \frac{x^6(-2+x^{-4}+x^{-6})}{x^8(4x^{-5}-5+6x^{-8})} = \lim_{x\to-\infty} \frac{1(-2+x^{-4}+x^{-6})}{x^2(4x^{-5}-5+6x^{-8})} = \lim_{x\to-\infty} \frac{-2}{-\infty\cdot(-5)} = 0$.

(f) $\lim_{x\to\infty} x^2 - \frac{x^4+1}{x^2+1}$
Solution: $\lim_{x\to\infty} x^2 - \frac{x^4+1}{x^2+1} = \lim_{x\to\infty} \frac{x^2(x^2+1)}{x^2+1} - \frac{x^4+1}{x^2+1} = \lim_{x\to\infty} \frac{x^4+x^2}{x^2+1} - \frac{x^4+1}{x^2+1} = \lim_{x\to\infty} \frac{x^4+x^2-(x^4+1)}{x^2+1} = \lim_{x\to\infty} \frac{x^4+x^2-x^4-1}{x^2+1} = \lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \lim_{x\to\infty} \frac{x^2(1-x^{-2})}{x^2(1+x^{-2})} = \lim_{x\to\infty} \frac{(1-x^{-2})}{(1+x^{-2})} = 1$.

(g) $\lim_{x\to\infty} \sqrt{x^2+1} - \sqrt{x^2-1}$
Solution: $\lim_{x\to\infty} \sqrt{x^2+1}-\sqrt{x^2-1} = \lim_{x\to\infty}(\sqrt{x^2+1}-\sqrt{x^2-1})\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{(x^2+1)-(x^2-1)}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{x^2+1-x^2+1}{\sqrt{x^2+1}+\sqrt{x^2-1}}$
$= \lim_{x\to\infty} \frac{2}{\sqrt{x^2+1}+\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{2}{x(\sqrt{1+x^{-2}}+\sqrt{1-x^{-2}})}$
$= \lim_{x\to\infty} \frac{1}{x} \lim_{x\to\infty} \frac{2}{\sqrt{1+x^{-2}}+\sqrt{1-x^{-2}}} = 0$.

(h) $\lim_{x\to\infty} \sqrt{x^2+1} - \sqrt{x-1}$
Solution:$\lim_{x\to\infty} \sqrt{x^2+1}-\sqrt{x-1} = \lim_{x\to\infty}(\sqrt{x^2+1}-\sqrt{x-1})\frac{\sqrt{x^2+1}+\sqrt{x-1}}{\sqrt{x^2+1}+\sqrt{x-1}} = \lim_{x\to\infty} \frac{(x^2+1)-(x-1)}{\sqrt{x^2+1}+\sqrt{x-1}} = \lim_{x\to\infty} \frac{x^2+1-x+1}{\sqrt{x^2+1}+\sqrt{x-1}}$
$= \lim_{x\to\infty} \frac{x^2-x+2}{\sqrt{x^2+1}+\sqrt{x-1}} = \lim_{x\to\infty} \frac{x^2(1-x^{-1}+2x^{-2})}{x(\sqrt{1+x^{-2}}+\sqrt{x^{-1}-x^{-2}})}$
$= \lim_{x\to\infty} \frac{x(1-x^{-1}+2x^{-2})}{(\sqrt{1+x^{-2}}+\sqrt{x^{-1}-x^{-2}})} = \lim_{x\to\infty} x \lim_{x\to\infty} \frac{(1-x^{-1}+2x^{-2})}{(\sqrt{1+x^{-2}}+\sqrt{x^{-1}-x^{-2}})} = \infty \cdot 1 = \infty$.

8. Find the domain of the following functions and determine the vertical and horizontal asymptotes of the graph of the following functions.

(a) $f(x) = \frac{x-1}{x^2-3x+2}$
Solution: To find the domain, we factor $x^2 - 3x + 2 = (x - 1)(x - 2)$. So the domain is $\{x|x \neq 1 \ and \ x \neq 2\}$. At $x = 1$, $\lim_{x\to 1^-} \frac{x-1}{x^2-3x+2} = \lim_{x\to 1^-} \frac{x-1}{(x-1)(x-2)} = \lim_{x\to 1^-} \frac{1}{(x-2)} = -1$ and $\lim_{x\to 1^+} \frac{x-1}{x^2-3x+2} = \lim_{x\to 1^+} \frac{x-1}{(x-1)(x-2)} = \lim_{x\to 1^+} \frac{1}{1(x-2)} = -1$. So $\lim_{x\to 1} \frac{x-1}{x^2-3x+2} = -1$. Hence $x = 1$ is not a vertical asymptote. In fact, $\frac{x-1}{x^2-3x+2}$ has a removable discontinuity at $x = 1$.
At $x = 2$, $\lim_{x\to 2^-} \frac{x-1}{x^2-3x+2} = \lim_{x\to 2^-} \frac{x-1}{(x-1)(x-2)} = \lim_{x\to 2^-} \frac{1}{(x-2)} = \frac{1}{0^-} = -\infty$ and $\lim_{x\to 2^+} \frac{x-1}{x^2-3x+2} = \lim_{x\to 2^+} \frac{x-1}{(x-1)(x-2)} = \lim_{x\to 2^+} \frac{1}{(x-2)} = \frac{1}{0^+} = \infty$. Hence $x = 2$ is a vertical asymptote.
$\lim_{x\to\infty} \frac{x-1}{x^2-3x+2} = \lim_{x\to\infty} \frac{x(1-x^{-1})}{x^2(1-3x^{-1}+2x^{-2})} \lim_{x\to\infty} \frac{1(1-x^{-1})}{x(1-3x^{-1}+2x^{-2})}$
$= \lim_{x\to\infty} \frac{1}{x} \lim_{x\to\infty} \frac{(1-x^{-1})}{(1-3x^{-1}+2x^{-2})} = 0{\cdot}1 = 0$. Similarly, $\lim_{x\to-\infty} \frac{x-1}{x^2-3x+2} = \lim_{x\to-\infty} \frac{x(1-x^{-1})}{x^2(1-3x^{-1}+2x^{-2})} \lim_{x\to-\infty} \frac{1(1-x^{-1})}{x(1-3x^{-1}+2x^{-2})}$
$= \lim_{x\to-\infty} \frac{1}{x} \lim_{x\to-\infty} \frac{(1-x^{-1})}{(1-3x^{-1}+2x^{-2})} = 0 \cdot 1 = 0$. Hence $y = 0$ is a horizontal asymptote.

(b) $f(x) = \frac{x-1}{x^2-5x+6}$
To find the domain, we factor $x^2 - 5x + 6 = (x - 2)(x - 3)$. So the domain is $\{x|x \neq 2 \ andx \neq 3\}$. At $x = 2$, $\lim_{x\to 2^-} \frac{x-1}{x^2-5x+6} = \lim_{x\to 2^-} \frac{x-1}{(x-2)(x-3)} = \frac{1}{0^-\cdot(-1)} = \frac{1}{0^+} = \infty$ and $\lim_{x\to 2^+} \frac{x-1}{x^2-5x+6} = \lim_{x\to 2^+} \frac{x-1}{(x-2)(x-3)} == \frac{1}{0^+\cdot(-1)} = \frac{1}{0^-} = -\infty$ Hence $x = 2$ is a vertical asymptote.
At $x = 3$, $\lim_{x\to 3^-} \frac{x-1}{x^2-5x+6} = \lim_{x\to 3^-} \frac{x-1}{(x-2)(x-3)} == \frac{2}{1\cdot 0^-} = \frac{2}{0^-} = -\infty$ and $\lim_{x\to 3^+} \frac{x-1}{x^2-5x+6} = \lim_{x\to 3^+} \frac{x-1}{(x-2)(x-3)} == \frac{2}{1\cdot 0^+} = \frac{2}{0^+} = \infty$. Hence $x = 3$ is a vertical asymptote.

$\lim_{x\to-\infty} \frac{x-1}{x^2-5x+6} = \lim_{x\to-\infty} \frac{x(1-x^{-1})}{x^2(1-5x^{-1}+6x^{-2})} \lim_{x\to-\infty} \frac{1(1-x^{-1})}{x(1-5x^{-1}+6x^{-2})}$
$= \lim_{x\to-\infty} \frac{1}{x} \lim_{x\to-\infty} \frac{(1-x^{-1})}{(1-5x^{-1}+6x^{-2})} = 0{\cdot}1 = 0$. Similarly, $\lim_{x\to\infty} \frac{x-1}{x^2-5x+6} = \lim_{x\to\infty} \frac{1}{x} \lim_{x\to\infty} \frac{(1-x^{-1})}{(1-5x^{-1}+6x^{-2})} = 0 \cdot 1 = 0$. Hence $y = 0$ is a horizontal asymptote.

(c) $f(x) = \frac{x^3-1}{x^2-5x+6}$
To find the domain, we factor $x^2 - 5x + 6 = (x - 2)(x - 3)$. So the domain is $\{x|x \neq 2 \ and \ x \neq 3\}$. At $x = 2$, $\lim_{x\to 2^-} \frac{x^3-1}{x^2-5x+6} = \lim_{x\to 2^-} \frac{x^3-1}{(x-2)(x-3)} = \frac{7}{0^-\cdot(-1)} = \frac{7}{0^+} = \infty$ and $\lim_{x\to 2^+} \frac{x^3-1}{x^2-5x+6} = \lim_{x\to 2^+} \frac{x^3-1}{(x-2)(x-3)} == \frac{7}{0^+\cdot(-1)} = \frac{7}{0^-} = -\infty$ Hence $x = 2$ is a vertical asymptote.
At $x = 3$, $\lim_{x\to 3^-} \frac{x^3-1}{x^2-5x+6} = \lim_{x\to 3^-} \frac{x^3-1}{(x-2)(x-3)} == \frac{26}{1\cdot 0^-} = \frac{26}{0^-} = -\infty$ and

$\lim_{x\to 3^+} \frac{x^3-1}{x^2-5x+6} = \lim_{x\to 3^+} \frac{x^3-1}{(x-2)(x-3)} == \frac{26}{1\cdot 0^+} = \frac{26}{0^+} = \infty$. Hence $x = 3$ is a vertical asymptote.

$\lim_{x\to -\infty} \frac{x^3-1}{x^2-5x+6} = \lim_{x\to -\infty} \frac{x^3(1-x^{-3})}{x^2(1-5x^{-1}+6x^{-2})} \lim_{x\to -\infty} \frac{x(1-x^{-3})}{(1-5x^{-1}+6x^{-2})}$ $= \lim_{x\to -\infty} x \lim_{x\to -\infty} \frac{(1-x^{-1})}{(1-5x^{-1}+6x^{-2})} = -\infty\cdot 1 = -\infty$. Similarly, $\lim_{x\to\infty} \frac{x^3-1}{x^2-5x+6} = \lim_{x\to\infty} x \lim_{x\to\infty} \frac{(1-x^{-1})}{(1-5x^{-1}+6x^{-2})} = \infty$. Hence $y = \frac{x^3-1}{(x-2)(x-3)}$ doesn't have a horizontal asymptote.

(d) $f(x) = \frac{-x^2+1}{x^2-5x+6}$

To find the domain, we factor $x^2 - 5x + 6 = (x-2)(x-3)$. So the domain is $\{x | x \neq 2 \textit{ and } x \neq 3\}$. At $x = 2$, $\lim_{x\to 2^-} \frac{-x^2+1}{x^2-5x+6} = \lim_{x\to 2^-} \frac{-x^2+1}{(x-2)(x-3)} == \frac{-3}{0^-\cdot(-1)} = \frac{-3}{0^+} = -\infty$ and $\lim_{x\to 2^+} \frac{-x^2+1}{x^2-5x+6} = \lim_{x\to 2^+} \frac{-x^2+1}{(x-2)(x-3)} = \frac{-3}{0^+\cdot(-1)} = \frac{-3}{0^-} = \infty$ Hence $x = 2$ is a vertical asymptote.

At $x = 3$, $\lim_{x\to 3^-} \frac{-x^2+1}{x^2-5x+6} = \lim_{x\to 3^-} \frac{-x^2+1}{(x-2)(x-3)} == \frac{-8}{1\cdot 0^-} = \frac{-8}{0^-} = \infty$ and $\lim_{x\to 3^+} \frac{-x^2+1}{x^2-5x+6} = \lim_{x\to 3^+} \frac{-x^2+1}{(x-2)(x-3)} == \frac{-8}{1\cdot 0^+} = \frac{-8}{0^+} = -\infty$. Hence $x = 3$ is a vertical asymptote.

$\lim_{x\to -\infty} \frac{-x^2+1}{x^2-5x+6} = \lim_{x\to -\infty} \frac{x^2(-1+x^{-2})}{x^2(1-5x^{-1}+6x^{-2})} \lim_{x\to -\infty} \frac{1(-1+x^{-2})}{(1-5x^{-1}+6x^{-2})}$ $= -1$. Similarly, $\lim_{x\to -\infty} \frac{-x^2+1}{x^2-5x+6} = \lim_{x\to -\infty} \frac{1(-1+x^{-2})}{(1-5x^{-1}+6x^{-2})} = -1$. Hence $y = -1$ is a horizontal asymptote.