Solution to Problem Set #3

1.(a)(30 pts) Change each of the following points from rectangular coordinates to cylindrical coordinates and spherical coordinates:

$$(2, 1, -2), (\sqrt{2}, 1, 1), (-2\sqrt{3}, -2, 3)\,.$$

Solution. $(2, 1, -2)$: Since $r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $\tan\theta = \frac{y}{x} = \frac{1}{2}$, the point in cylindrical coordinates is $\left(\sqrt{5}, \arctan\left(\frac{1}{2}\right), -2\right)$. Similarly,

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + 1^2 + (-2)^2} = 3\,,$$

$\tan\theta = \frac{y}{x} = \frac{1}{2}$ and $\cos\phi = \frac{z}{\rho} = \frac{-2}{3}$ so the point in spherical coordinates is $\left(3, \arctan\left(\frac{1}{2}\right), \frac{\pi}{2} + \arccos\left(\frac{\sqrt{5}}{3}\right)\right)$ or $\left(3, \arctan\left(\frac{1}{2}\right), \frac{\pi}{2} + \arctan\left(\frac{2}{\sqrt{5}}\right)\right)$.

$(\sqrt{2}, 1, 1)$: Since $r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$ and $\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{2}}$, the point in cylindrical coordinates is $\left(\sqrt{3}, \arctan\left(\frac{1}{\sqrt{2}}\right), 1\right)$. Similarly,

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(\sqrt{2})^2 + 1^2 + 1^2} = 2\,,$$

$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{2}}$ and $\cos\phi = \frac{z}{\rho} = \frac{1}{2}$ so the point in spherical coordinates is $\left(2, \arctan\left(\frac{1}{\sqrt{2}}\right), \frac{\pi}{3}\right)$.

$(-2\sqrt{3}, -2, 3)$: Since $r = \sqrt{x^2 + y^2} = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = 4$ and

$$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}}\,,$$

the point in cylindrical coordinates is $\left(4, \frac{7\pi}{6}, 3\right)$. Similarly,

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-2\sqrt{3})^2 + (-2)^2 + 3^2} = 5\,,$$

$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}}$ and $\cos\phi = \frac{z}{\rho} = \frac{3}{5}$ so the point in spherical coordinates is $\left(5, \frac{7\pi}{6}, \arccos\left(\frac{3}{5}\right)\right)$ or $\left(5, \frac{7\pi}{6}, \arctan\left(\frac{4}{3}\right)\right)$. $\square$

1.(b) (15 pts) Convert the equation $r^2\cos(2\theta) = z^2$ into rectangular coordinates.

Solution. Since $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$r^2\cos(2\theta) = r^2\big(\cos^2(\theta) - \sin^2(\theta)\big) = \big(r\cos(\theta)\big)^2 - \big(r\sin(\theta)\big)^2$$

which implies that the equation $r^2\cos(2\theta) = z^2$ is $x^2 - y^2 = z^2$ in rectangular coordinates. □

1.(c)(15 pts) Convert the equation $\rho\sin(\phi) = 1$ into rectangular coordinates.

Solution. Since we have

$$\big(\rho\sin(\phi)\big)^2 = \rho^2\sin^2(\phi)\big(\cos^2(\theta) + \sin^2(\theta)\big) = \big(\rho\sin(\phi)\cos(\theta)\big)^2 + \big(\rho\sin(\phi)\sin(\theta)\big)^2,$$

the equation $\rho\sin(\phi) = 1$ is $\sqrt{x^2+y^2} = 1$ or $x^2 + y^2 = 1$in rectangular coordinates. Note that we have used $x = \rho\sin(\phi)\cos(\theta)$ and $y = \rho\sin(\phi)\sin(\theta)$. □

2(20 pts) Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

Solution. We can parameterize the cylinder $x^2 + y^2 = 9$ by $(3\cos(t), 3\sin(t), z)$. Now $x = 3\cos(t)$ and $y = 3\sin(t)$. So $z = xy = 3\cos(t)\cdot 3\sin(t) = 9\cos(t)\sin(t)$. So the intersection of the cylinder $x^2 + y^2 = 9$ and the surface $z = xy$ can be represented by $(3\cos(t), 3\sin(t), 9\cos(t)\sin(t))$. □

3(20 pts) Find a vector function that represents the curve of intersection of the cone $z = \sqrt{x^2+y^2}$ and the plane $z = x + 2$.

Solution. Both equations ($z = \sqrt{x^2+y^2}$ and $z = x + 2$) are solved for z, so we can substitute to eliminate z: $\sqrt{x^2+y^2} = x + 2 \Rightarrow x^2 + y^2 = (x+2)^2 = x^2 + 4x + 4$ $\Rightarrow x = \frac{y^2}{4} - 1$. We can determine the parametric equation for the curve C of intersection by choosing $y = t$, then $x = \frac{t^2}{4} - 1$ and $z = x + 2 = \frac{t^2}{4} - 1 + 2 = \frac{t^2}{4} + 1$. Thus a vector function representation is $(\frac{t^2}{4} - 1, t, \frac{t^2}{4} + 1)$.

□