Math 6(8)540 Homework 4 Due date: November 14

(1) Verify that if $n > 1$, the unbounded function $u = \ln\ln(1 + \frac{1}{|x|})$ belongs to $W^{1,n}(B(0,1))$, i.e. $u, Du, \in L^n(B(0,1))$.

(2) Assume $F : R \mapsto R$ is C^1, with F' bounded. Suppose that Ω is bounded and $u \in W^{1,p}(\Omega)$ for some $1 < p < \infty$. Show that $v := F(u) \in W^{1,p}(\Omega)$ and $v_{x_i} = F'(u)u_{x_i}$ (in the sense of weak partial derivative.) $(i = 1, \cdots, n)$.

(3) Assume $1 < p < \infty$ and Ω is bounded .

(a) Prove that if $u \in W^{1,p}(\Omega)$ for then $|u| \in W^{1,p}(\Omega)$.

(b) Prove that if $u \in W^{1,p}(\Omega)$ implies u^+, $u^- \in W^{1,p}(\Omega)$, and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\} \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \geq 0\} \\ -Du & \text{a.e. on } \{u < 0\} \end{cases}$$

(Hint:$u^+ = \lim_{\epsilon \to 0} F_\epsilon(u)$ where $F_\epsilon(z) := \begin{cases} (z^2 + \epsilon^2) - \epsilon & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$)

(c) Prove that if $u \in W^{1,p}(\Omega)$ then $Du = 0$ a.e. on the set $\{u = 0\}$.