Math 6(8)540 Homework 3 Due date: October 25

(1) In the class, we have proved that $\frac{||y-\frac{x}{||x||^2}||^2}{||y-x||^2} = \frac{1}{||x||^2}$ for $||y|| = 1$. Consider the quantity $f(y) = \frac{||y-ax||^2}{||y-x||^2}$. Show that $f(y)$ is independent of y for $||y|| = 1$ if and only if $a = 1$ or $a = \frac{1}{||x||^2}$. (Hint: Just consider the two dimensional case and parameterize S^1 by $y = (\cos(t), \sin(t))$.)

(2) Let $G(x, y)$ be he Green's function on Ω. Use the strong maximum principle to prove that the Green's function $G(x, y) > 0$ for $x \neq y$. (Hint: Consider the function $u(z) = G(x, z) = \Phi(z - x) - \phi^x(z)$ on $\Omega \setminus B(x, \epsilon)$.)

(3) (a) Suppose $\triangle u = 0$ in Ω and $u|_{\partial\Omega} = g$. Show that the Poisson's formula on the unit disk in R^2

$$u(x) = \frac{1 - |x|^2}{2\pi} \int_{S^1} \frac{g(y)}{|x - y|^2} dS(y)$$

is equivalent to

$$u(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{g(\phi)}{1 + r^2 - 2r\cos(\theta - \phi)} d\phi.$$

(b) Use formula in part (a) to prove that

$$r^k \cos(k\theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{\cos(k\phi)}{1 + r^2 - 2r\cos(\theta - \phi)} d\phi$$

and

$$r^k \sin(k\theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{\sin(k\phi)}{1 + r^2 - 2r\cos(\theta - \phi)} d\phi$$

(4) Let $\mathfrak{A} = \{u | u \in C^2(\overline{\Omega}), u = g \text{ on } \partial\Omega\}$ Consider the area functional of the graph of u defined by $F(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx$. Show that the minimizer of the functional in $\mathfrak{A}$ satisfies

$$div(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0.$$

(Hint: Consider $f(\tau) = F(u + \tau\rho)$ where $\rho \in C_0^\infty(\Omega)$.)