Math 6(8)540 Homework 2 Due date: October 11

(1) A function $u \in L^1_{loc}(\Omega)$ is weakly harmonic if $\int_\Omega u \triangle \psi = 0$ for all $\psi \in C_c^\infty(\Omega)$. In the following. $u^\epsilon = \eta_\epsilon * u$ is the mollification of u

  (a) Prove that if $u \in L^1_{loc}(\Omega)$ is a weakly harmonic function then $\triangle u^\epsilon = 0$ on Ω_ϵ.

  (b) Prove that $u^\sigma = (u^\sigma)^\tau$ in $\Omega_{\sigma+\tau}$.

  (c) Prove that if $u \in L^1_{loc}(\Omega)$ is a weakly harmonic function then $u = v$ a.e. where v is a $C^\infty(\Omega)$ harmonic function. (Hint: Use the fact that $\lim_{\epsilon \to 0} f^\epsilon = f$ a.e. and consider $\lim_{\sigma \to 0} u_\sigma$).

(2) Recall that $u \in C^2(\overline{\Omega})$ is subharmonic if $\triangle u \geq 0$ in Ω. Now we want to give another notion of subharmonic function for continuous function. A $C^0(\Omega)$ function u is subharmonic in Ω if for every ball $B \subset\subset \Omega$ and every function h harmonic in B satisfying $u \leq h$ on ∂B, we also have $u \leq h$ in B. Prove that a $C^0(\Omega)$ subharmonic function satisfies the strong maximum principle. (Hint: Prove by contradiction. Suppose $x_0 \in \Omega$ where $u(x_0) = sup_\Omega u$. Consider h where $\triangle h = 0$ and $h = u$ on $\partial(B(x_0, r))$.

(3) Suppose f is defined in R^n, $\triangle f = 0$ and $|\nabla f| = 1$. Prove that f is a linear function. (Hint: Use Bochner formula).

(4) Suppose $u \in C^2(\overline{B_{2r}}) > 0$ satisfies $\triangle u = -\lambda u$ where λ is a positive constant.

  (a) Let $v = \ln u$ and $w = |\nabla v|^2$. Show that $\triangle v = -w - \lambda$.

  (b) Show that

$$\triangle(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4)$$
$$= 2\phi^4 |Hessv|^2 + 4 \sum_{ij} \frac{\partial \phi^4}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} + w\triangle\phi^3 + 2\nabla v \cdot \phi^4.$$

  (c) Prove that $sup_{B_r} \frac{|\nabla\ u|}{u} \leq C$ where C depends on n only.

(5) Use poisson formula for the ball to prove

$$r^{n-2} \frac{r - |x|}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq r^{n-2} \frac{r + |x|}{(r - |x|)^{n-1}} u(0)$$

whenever u is positive and harmonic in $B(0, r)$.