PDE NOTES
09/22/08

Gradient Estimate

Th: There are dimensional constants $C = C(n)$ such that

$$\sup_{B_r} \frac{|\nabla u|}{u} \le \frac{C}{r}$$

for all postive harmonic ftns $u : B_{2r} \to \mathbb{R}$

Maximum Principle for C^2 ftns:

$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ where Ω is bounded.

Suppose $u(x_0) = \max_{\overline{\Omega}} u$ where $x_0 \in \Omega$

(i.e. u achieves an interior maximum)

$$\Rightarrow \begin{cases} \nabla u(x_0) = 0 \\ \Delta u(x_0) \le 0 \end{cases}$$

$(u_{ij}) \le 0$ at x_0

(It follows from 2nd derivative test

$\Rightarrow$ Hess $(u)(x_0) = (\, u_{ij}(x_0)) \le 0$ nonpositive definite

and $\Delta u(x_0) = \operatorname{trace}((u_{ij}(x_0))) \le 0$

(Bochner formula)

Prop: Let $u \in C^2(\Omega)$.

Then $\frac{1}{2} \Delta |\nabla u|^2 = \langle \nabla \Delta u, \nabla u \rangle + |\mathrm{Hess}\, u|^2$

where $\langle \ , \ \rangle$ is the inner product in $\mathbb{R}^n$

$|\mathrm{Hess}\, u|^2 = \sum_{i,j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2$ and

$\mathrm{Hess}(u) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n}$

pf: $|\nabla u|^2 = \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} \right)^2$

$$
\begin{aligned}
\Delta |\nabla u|^2 &= \sum_i \frac{\partial^2}{\partial x_i^2} \left(\sum_j \left(\frac{\partial u}{\partial x_j} \right)^2 \right) \\
&= \sum_i \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_i} \left(\sum_j \left(\frac{\partial u}{\partial x_j} \right)^2 \right) \\
&= \sum_{i,j} \frac{\partial}{\partial x_i} \left(2 \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_j} \right) \\
&= \sum_{i,j} \left(2 \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial u}{\partial x_j} + 2 \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \\
&= \sum_{i,j} 2 \frac{\partial}{\partial x_j} \left(\frac{\partial^2 u}{\partial x_i^2} \right) \cdot \frac{\partial u}{\partial x_j} + 2 |\mathrm{Hess}(u)|^2 \\
&= 2 \sum_j \frac{\partial}{\partial x_j} (\Delta u) \cdot \frac{\partial u}{\partial x_j} + 2 |\mathrm{Hess}(u)|^2 \\
&= 2 \langle \nabla \Delta u, \nabla u \rangle + 2 |\mathrm{Hess}(u)|^2
\end{aligned}
$$

$\Rightarrow \frac{1}{2} \Delta |\nabla u|^2 = \langle \nabla \Delta u, \nabla u \rangle + |\mathrm{Hess}(u)|^2$

Remark: On a Riemannian manifold

We can define ∇u and Δu

$\exists$ $\frac{1}{2} \Delta |\nabla u|^2 = \langle \nabla \Delta u, \nabla u \rangle + |\mathrm{Hess}(u)|^2 + \underline{\mathrm{Ric}}(\nabla u, \nabla u)$

Ricci Curvature.

Pf of the gradient Estimate:

We prove the result for $r=1$ 1st

Choose $\phi : B_{2r} \longrightarrow \mathbb{R}$ with $\phi = 0$ on ∂B_{2r}.

Define $\begin{cases} v = \log(u) & \left(b/c\ u>0 \Rightarrow \ln(u) \text{ is well-defined} \right) \\ w = |\nabla v|^2 \left(= \frac{|\nabla u|^2}{u^2} \right) \end{cases}$

We want to estimate w.

Note that $\nabla v = \nabla (\ln(u)) = \frac{\nabla u}{u}$

$$\Delta v = \operatorname{div}(\nabla v) = \operatorname{div}\left(\frac{\nabla u}{u} \right) = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}$$

$$= -\frac{|\nabla u|^2}{u^2} \quad \left(b/c\ \Delta u = 0 \right)$$

$$= -w$$

Compute $\Delta w = \Delta (|\nabla v|^2)$

$$= 2\langle \nabla \Delta v, \nabla v\rangle + 2|\mathrm{Hess}\, v|^2$$

$$= -2\langle \nabla w, \nabla v\rangle + 2|\mathrm{Hess}(v)|^2$$

$$(\Delta v = -w)$$

Compute $\Delta(w\phi^4)$

$$= (\Delta w)\phi^4 + 2\langle \nabla w, \nabla \phi^4\rangle + w\Delta\phi^4$$

$$\left(\Delta(fg) = \Delta f\, g + 2\langle \nabla f, \nabla g\rangle + f\Delta g\right)$$

$$= 2\phi^4|\mathrm{Hess}(v)|^2 - 2\phi^4\langle \nabla w, \nabla v\rangle + 2\, \nabla w \cdot \nabla\phi^4 + w\Delta\phi^4$$

Note that $\langle \nabla(\phi^4 w), \nabla v\rangle$

$$= w\langle \nabla\phi^4, \nabla v\rangle + \phi^4\langle \nabla w, \nabla v\rangle$$

$$\Rightarrow \quad -\phi^4\langle \nabla w, \nabla v\rangle = -\langle \nabla(\phi^4 w), \nabla v\rangle + w\langle \nabla\phi^4, \nabla v\rangle$$

$$\Rightarrow \quad \Delta(w\phi^4) = 2\phi^4|\mathrm{Hess}(v)|^2 - 2\langle \nabla(\phi^4 w), \nabla v\rangle + 2w\langle \nabla\phi^4, \nabla v\rangle + 2\langle \nabla w, \nabla\phi^4\rangle + w\Delta\phi^4$$

$$\Rightarrow \quad \Delta(w\phi^4) + 2\langle \nabla(\phi^4 w), \nabla v\rangle$$

$$= 2\phi^4|\mathrm{Hess}(v)|^2 + 2\langle \nabla\phi^4, \nabla w\rangle + w\Delta\phi^4 + 2w\langle \nabla v, \nabla\phi^4\rangle$$

Notes will be available at
www.math.utoledo.edu/~mtsui

$\Rightarrow$ Link to this class

$\Rightarrow$ HW and schedule.

Th: $\displaystyle \sup_{B_r} \frac{|\nabla u|}{u} \le \frac{C}{r}$

for all harmonic ftns $u : B_{2r} \to \mathbb{R}$.

Last time, we showed that

$$\left(\begin{array}{l} \phi : B_2 \to \mathbb{R} \text{ with } \phi\big|_{\partial B_2} = 0, \\ \qquad\qquad\qquad\quad \phi > 0 \text{ in } B_2 \end{array} \right)$$

$$v = \ln u, \quad w = |\nabla v|^2 \left(= \frac{|\nabla u|^2}{u^2} \right)$$

$$\Delta(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4)$$

$$= 2\phi^4 |\mathrm{Hess}(v)|^2 + 2\nabla\phi^4 \cdot \nabla w + w\Delta\phi^4 + 2w\nabla v \cdot \nabla\phi^4$$

Compute $2\nabla\phi^4 \cdot \nabla w$

$$= \sum_i 2 \frac{\partial}{\partial x_i}(\phi^4) \cdot \frac{\partial}{\partial x_i} \overbrace{\left(\sum_j \left(\frac{\partial v}{\partial x_j} \right)^2 \right)}^{w \,=}$$

$$= 4 \sum_{i,j} \frac{\partial}{\partial x_i}(\phi^4) \frac{\partial v}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$$

$$\Rightarrow \quad \Delta(w\phi^4) + 2\nabla V \cdot \nabla(w\phi^4)$$

$$= \sum_{i,j} 2\phi^4\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)^2 + 4\frac{\partial \phi^4}{\partial x_i}\frac{\partial V}{\partial x_j}\frac{\partial^2 V}{\partial x_i \partial x_j} + w\Delta\phi^4$$
$$+ 2w\nabla V \cdot \nabla\phi^4$$

Let $a_{ij} = \phi^2 \frac{\partial^2 V}{\partial x_i \partial x_j}$ and $b_{ij} = \frac{\partial \phi^4}{\partial x_i} \cdot \frac{\partial V}{\partial x_j}$

$$\begin{cases} a_{ij}^2 + 4a_{ij}b_{ij} \geq -4b_{ij}^2 \\ \sum_{i,j} b_{ij}^2 = |\nabla\phi^4|^2 |\nabla V|^2 \end{cases}$$

$$\left(\sum_{i,j} b_{ij}^2 = \sum_{i,j}\left(\frac{\partial\phi^4}{\partial x_i}\right)^2\left(\frac{\partial V}{\partial x_j}\right)^2 = \sum_i\left(\frac{\partial\phi^4}{\partial x_i}\right)^2 \cdot \sum_j\left(\frac{\partial V}{\partial x_j}\right)^2 \right.$$
$$= |\nabla\phi^4|^2 |\nabla V|^2$$

$$\Delta(w\phi^4) + 2\nabla V \cdot \nabla(w\phi^4)$$
$$\Rightarrow \quad = \sum_{i,j} \phi^4\left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)^2 + \quad a_{ij}^2 + 4a_{ij}b_{ij} + w\Delta\phi^4 + 2w\nabla V \cdot \nabla\phi^4$$

$$\Delta(w\phi^4) + 2\nabla v\cdot\nabla(w\phi^4)$$

$$\Rightarrow = \sum_{i,j} \phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 + a_{ij}^2 + 4a_{ij}b_{ij} + w\Delta\phi^4 + 2w\nabla v\cdot\nabla\phi^4$$

$$\geq \sum_{i,j} \phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 4b_{ij}^2 + w\Delta\phi^4 + 2w\nabla v\cdot\nabla\phi^4$$

$$= \sum_{i,j} \phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 4|\nabla\phi^4|^2|\nabla v|^2 + w\Delta\phi^4 + 2w\nabla v\cdot\nabla\phi^4$$

$$\nabla\phi^4 = 4\phi^3\nabla\phi, \quad \nabla v\cdot\nabla\phi^4 = 4\phi^3\nabla v\cdot\nabla\phi$$

$$\Delta\phi^4 = \operatorname{div}(\nabla\phi^4) = \operatorname{div}(4\phi^3\cdot\nabla\phi)$$

$$= 12\phi^2|\nabla\phi|^2 + 4\phi^3\Delta\phi$$

$$|\nabla\phi^4|^2 = 16\phi^6|\nabla\phi|^2$$

$$= \sum_{i,j} \phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 64\phi^6|\nabla\phi|^2|\nabla v|^2$$

$$+ 12w\phi^2|\nabla\phi|^2 + 4w\phi^3\Delta\phi + \underbrace{8w\phi^3\nabla v\cdot\nabla\phi}_{\geq}$$

$$\geq \sum_{i,j} \phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 64\phi^4|\nabla\phi|^2|\nabla v|^2$$

$$+ 12w\phi^2|\nabla\phi|^2 + 4w\phi^3\Delta\phi - \underline{8w\phi^3|\nabla v||\nabla\phi|}$$

$$= \sum_{i,j} \phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 64\phi^6|\nabla\phi|^2|\nabla v|^2$$

$$+ \left(12\phi^2|\nabla\phi|^2 + 4\phi^3\Delta\phi\right)w - 8\phi^3 w|\nabla v||\nabla\phi|$$

b/c ϕ is a $C^2(\overline{B_{2r}})$, $\exists\ c_1>0, c_2>0, c_3>0$

$$\Rightarrow -64\phi^4|\nabla\phi|^2 \geq -c_1$$

$$\Rightarrow 4\phi\Delta\phi + 12|\nabla\phi|^2 \geq -c_2$$

$$-8|\nabla\phi| \geq -c_3$$

$$\geq \sum_{i,j} \phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - c_1\phi^2|\nabla v|^2 - c_2\phi^2 w - c_3\phi^3 w|\nabla v|$$

$$\frac{\sum_{i,j}\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2}{n} \geq \frac{\sum_{i=1}^n\left(\frac{\partial^2 v}{\partial x_i^2}\right)^2}{n} \geq \left(\frac{\sum_i\frac{\partial^2 v}{\partial x_i^2}}{n}\right)^2$$

$$= \left(\frac{\Delta v}{n}\right)^2 n$$

$$\geq \phi^4\frac{(\Delta v)^2}{n} - c_1\phi^2|\nabla v|^2 - c_2\phi^2 w - c_3\phi^3 w|\nabla v|$$

(Recall $\Delta v = -w$, $|\nabla v|^2 = w$

$$\Rightarrow \Delta(w\phi^4) + 2\nabla V \cdot \nabla(w\phi^4)$$

$$* \left(\geq \frac{\phi^4}{n} w^2 - c_1\phi^2 w - c_2\phi^2 w - c_3\phi^3 w^{\frac{3}{2}} \right.$$

b/c $w > 0$ and $\phi > 0$ in B_{2r} ($\phi = 0$ on ∂B_{2r}).

$$\begin{cases} w\phi^4 > 0 \text{ in } B_{2r} \\ w\phi^4 = 0 \text{ on } \partial B_{2r} \end{cases}$$

b/c $w\phi^4 \in C^2(\overline{B_2}) \Rightarrow w\phi^4$ must achieve an interior maximum at x_0

$$\Rightarrow \begin{cases} \Delta(w\phi^4)(x_0) \leq 0 \\ \nabla(w\phi^4)(x_0) = 0 \end{cases}$$

PDE Notes:
09/26/08
6pages

Recall
$$\begin{cases} v = \ln u \\ w = |\nabla v|^2 = \dfrac{|\nabla u|^2}{u^2} \geq 0 \\ \phi : B_2 \longrightarrow \mathbb{R} \\ \phi > 0 \text{ in } B_2 \text{ and } \phi\big|_{\partial B_2} = 0 \end{cases}$$

From last time, we derived

$$\Delta(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4)$$
$$\geq \frac{\phi^4 w^2}{n} - c_1\phi^2 w - c_2\phi^2 w - c_3\phi^3 w^{\frac{3}{2}}$$

[where c_1, c_2, c_3 depends on ϕ. (not on w)]

$$\| \quad \phi^2 w\left[\frac{\phi^2 w}{n} - c_3\phi w^{\frac{1}{2}} - (c_1+c_2)\right]$$

Since $\begin{cases} w\phi^4 \geq 0 \text{ in } B_2 \\ w\phi^4\big|_{\partial B_2} = 0 \quad (\text{b/c } \phi\big|_{\partial B_2} = 0) \end{cases}$,

We can find $x_0 \in B_2$ s.t.
$$(w\phi^4)(x_0) = \max_{B_2} w\phi^4$$

$$\Rightarrow \begin{cases} \nabla(w\phi^4)(x_0) = 0 \\ \Delta(w\phi^4)(x_0) \leq 0 \end{cases}$$

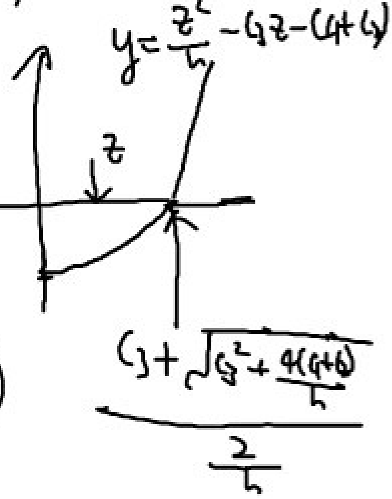
$$\Rightarrow \quad \phi^2 w(x_0)\left[\frac{\phi^2 w(x_0)}{n} - c_3\phi w^{\frac{1}{2}}(x_0) - (c_1+c_2)\right] \leq 0$$

$\Rightarrow$ Let $z = (\phi w^{\frac{1}{2}})(x_0) \geq 0$ b/c $\phi^2 w(x_0) \geq 0$

$$\Rightarrow \quad \frac{z^2}{n} - c_3 z - (c_1+c_2) \leq 0$$

The roots of $\dfrac{x^2}{n} - c_3 x - (c_1+c_2) = 0$

are $x = \dfrac{c_3 \pm \sqrt{c_3^2 + \frac{4(c_1+c_2)}{n}}}{\frac{2}{n}}$

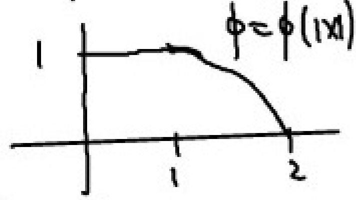


$$z \leq \frac{c_3 + \sqrt{c_3^2 + 4\frac{(c_1+c_2)}{n}}}{\frac{2}{n}} = C(n)$$

$$\Longrightarrow \quad (\phi^2 w)(x_0) \le C(n)$$

where $\phi^4 w(x_0) = \sup_{B_2} \phi^4 w$

Require $0 \le \phi \le 1$ and $\phi \equiv 1$ in B_1



$$\sup_{B_2} \phi^4 w = \phi^4 w(x_0)$$
$$\le \phi^2 w(x_0) \quad (\text{b/c } 0 \le \phi \le 1)$$
$$\le C(n)$$

Moreover, $\sup_{B_1} \phi^4 w \le \sup_{B_2} \phi^4 w \le C(n)$

$$\sup_{B_1} \phi^4 w = \sup_{B_1} w \quad (\text{b/c } \phi \equiv 1 \text{ in } B_1)$$

Recall that $w = \frac{|\nabla u|^2}{u^2}$

$$\Longrightarrow \quad \sup_{B_1} \frac{|\nabla u|^2}{u^2} \le C(n)$$

$$\Longrightarrow \quad \sup_{B_1} \frac{|\nabla u|}{u} \le \sqrt{C(n)}$$

So we have proved that

$$\begin{cases} u: B_2 \to \mathbb{R} \text{ and harmonic} \\ u > 0 \end{cases}$$

$$\Rightarrow \quad \frac{|\nabla u|}{u} \le C(n) \text{ in } B_1$$

Now we consider the general case.

$$u: B_{2r} \to \mathbb{R}, \ u > 0, \text{ harmonic}.$$

Let $v(x) = u(rx)$.

Then $v: B_2 \to \mathbb{R}$, $v > 0$, harmonic

$$\Rightarrow \quad \frac{|\nabla v(x)|}{v(x)} \le C(n), \ x \in B_1$$

Compute $\nabla v(x) = r \nabla u(rx)$

$$\Rightarrow \quad \frac{|\nabla v(x)|}{v(x)} = \frac{r|\nabla u(rx)|}{u(rx)} \le C(n)$$

$$\Rightarrow \quad \frac{|\nabla u(rx)|}{u(rx)} \le \frac{C(n)}{r}$$

$$\Rightarrow \quad \frac{|\nabla u(y)|}{u(y)} \le \frac{C(n)}{r}, \ y \in B_r$$

The techniques used in this proof can be used to study other PDE.

For example: $\Delta u = 0$ on a Riemann mfld with $Rc \geq 0$

Minimal surface eqs.

Gradient estimate of this type ($\frac{|\nabla u|}{u} \leq \frac{C}{r}$) holds for other eqs.

$\Downarrow$ This implies

Harnack inequality.

§2.2.4 Green's fct

Assume $\Omega \subset \mathbb{R}^n$ is open, bounded and $\partial\Omega$ is C^1 (so the normal is well-defined)

In the following, we'll derive a representation formula for the sol of
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Th: (Green's representation formula)

$u \in C^2(\bar{\Omega})$

$$\Rightarrow u(x) = \int_{\partial\Omega} \left(\Phi(y-x) \frac{\partial u(y)}{\partial \nu} - u(y) \frac{\partial \Phi}{\partial \nu}(y-x) \right) dS(y)$$

$$- \int_{\Omega} \Phi(y-x) \Delta u(y) dy$$

for any $x \in \Omega$

where $\Phi(x) \underset{\uparrow}{=\!=} \begin{cases} -\frac{1}{2\pi} \ln|x| \\ \frac{1}{n(n-2)\alpha_n} |x|^{2-n} \end{cases}$

fundamental sol

Remark ① If $\Delta u = 0$ in Ω

$$\Rightarrow u(x) = \int_{\partial\Omega} \left(\Phi(y-x) \frac{\partial u(y)}{\partial \nu} - \underline{u(y)} \frac{\partial \Phi}{\partial \nu}(y-x) \right) dS(y)$$

Disadvantage This formula depends on $\frac{\partial u}{\partial \nu}$

This is unknown.

Boundary value (known)

(Pf of Green's representation formula)

PDE NOTES
on Sep 29
3 pages

Given $x \in \Omega$,

Choose $\varepsilon > 0$ small enough so that

$$B(x,\varepsilon) \subset \Omega$$

Note that $\Phi(y-x)$ is not defined when $y = x$.

Apply Green's 2-nd Identity to

$u(y)\,\Phi(y-x)$ (regard this as a fcn of y)

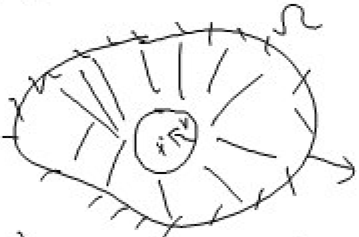
in $\Omega \setminus B(x,\varepsilon)$

$$\int_{\Omega \setminus B(x,\varepsilon)} \left(u(y)\,\Delta_y \Phi(y-x) - \Phi(y-x)\,\Delta_y u(y) \right) dy$$

$$= \int_{\partial(\Omega \setminus B(x,\varepsilon))} \left[u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u(y)}{\partial \nu} \right] dS(y)$$

Recall that Φ is a fundamental sol

$\Rightarrow \quad \Delta_y \Phi(y-x) = 0$



$$\partial(\Omega \setminus B(x,\varepsilon)) = \partial\Omega \cup \partial(B(x,\varepsilon))$$

$$\Rightarrow \int_{\partial(\Omega \setminus B(x,\varepsilon))} \left[u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u(y)}{\partial \nu} \right] dS(y)$$

$$= \int_{\partial\Omega} \left[u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u(y)}{\partial \nu} \right] dS(y)$$

$$+ \int_{\partial(B(x,\varepsilon))} \left[u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u(y)}{\partial \nu} \right] dS(y)$$

Claim $\displaystyle\int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi(y-x)}{\partial \nu}\,dS(y) = \frac{1}{n\alpha_n \varepsilon^{n-1}}\int_{\partial B(x,\varepsilon)} u(y)\,dS(y)$

Recall that $\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| \\ \frac{1}{n(n-2)\alpha_n}|x|^{2-n} \end{cases}$

$$\Rightarrow \nabla\Phi(x) = \begin{cases} -\frac{1}{2\pi|x|}\frac{x}{|x|} \\ -\frac{1}{n\alpha_n}|x|^{1-n}\frac{x}{|x|} \end{cases}$$

$$\Rightarrow \nabla_y\Phi(y-x) = \begin{cases} -\frac{1}{2\pi|y-x|}\frac{y-x}{|y-x|} \\ -\frac{1}{n\alpha_n}|y-x|^{1-n}\frac{y-x}{|y-x|} \end{cases}$$

On $\partial B(x,\varepsilon)$

$\nu = -\frac{(y-x)}{\|y-x\|}$

$$\Rightarrow \frac{\partial\Phi(y-x)}{\partial\nu} = \nabla_y\Phi(y-x)\cdot\nu$$

$$= \begin{cases} -\frac{1}{2\pi|y-x|}\cdot\frac{y-x}{|y-x|}\cdot\left(-\frac{(y-x)}{|y-x|}\right) & n=2 \\ -\frac{1}{n\alpha_n}|y-x|^{1-n}\frac{y-x}{|y-x|}\cdot\left(-\frac{(y-x)}{|y-x|}\right) & n\geq 3 \end{cases}$$

$$= \begin{cases} \frac{1}{2\pi|y-x|} & n=2 \\ \frac{1}{n\alpha_n|y-x|^{n-1}} & n\geq 3 \end{cases}$$

$y\in\partial B(x,\varepsilon) \Rightarrow \|y-x\| = \varepsilon$

$$\Rightarrow \frac{\partial\Phi(y-x)}{\partial\nu} = \begin{cases} \frac{1}{2\pi\varepsilon} & n=2 \\ \frac{1}{n\alpha_n\varepsilon^{n-1}} & n\geq 3 \end{cases}$$

$$\Rightarrow \int_{\partial B(x,\varepsilon)} u(y)\frac{\partial\Phi(y-x)}{\partial\nu}\,dS(y) = \frac{1}{n\alpha_n\varepsilon^{n-1}}\int_{\partial B(x,\varepsilon)} u(y)\,dS(y)$$

Claim: $\displaystyle\left|\int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial u}{\partial\nu}\,dS(y)\right| \leq \begin{cases} C\varepsilon|\ln\varepsilon| & n=2 \\ C\varepsilon & n\geq 3 \end{cases}$

Note that $|\Phi(y-x)| = \begin{cases} C|\ln\varepsilon| \\ \frac{C}{\varepsilon^{n-2}} \end{cases}$ where $y\in\partial B(x,\varepsilon)$ $(|y-x|=\varepsilon)$

b/c $u\in C^2(\overline{\Omega}) \Rightarrow |Du|\leq C$ in $\overline{\Omega}$

$$\left|\frac{\partial u}{\partial \nu}\right| = |\nabla_y u \cdot \nu| \leq |\nabla_y u| \cdot |\nu|$$
$$\leq C \quad (\text{b/c } |\nu| = 1)$$

$$\Rightarrow \left| \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu} \, dS(y) \right|$$

$$\leq \int_{\partial B(x,\varepsilon)} |\Phi(y-x)| \left|\frac{\partial u}{\partial \nu}\right| dS(y)$$

$$\leq C \int_{\partial B(x,\varepsilon)} |\Phi(y-x)| \, dS(y)$$

$$\leq C \begin{cases} \varepsilon \, |\ln \varepsilon| & n=2 \\ \varepsilon^{n-1} \cdot \frac{1}{\varepsilon^{n-2}} & n \geq 3 \end{cases}$$

$$= \begin{cases} C \varepsilon \, |\ln \varepsilon| & n=2 \\ C \varepsilon & n \geq 3 \end{cases}$$

Combining previous two claims:

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi(y-x)}{\partial \nu} \, dS(y)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{n \alpha_n \varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} u(y) \, dS(y)$$

$$= u(x) \quad (\text{b/c } u \text{ is cts})$$

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \, dS(y)$$

$$= 0$$

Let $\varepsilon \longrightarrow 0$

$$\int_\Omega -\Phi(y-x) \Delta_y u(y) \, dy$$

$$= u(x) + \int_{\partial \Omega} \left[u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u}{\partial \nu} \right] dS(y)$$

$$u(x) = \int \left[\Phi(y-x) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial \Phi(y-x)}{\partial \nu} \right] dS(y) - \int_\Omega \Phi(y-x) \Delta u(y) \, dy$$

↑ Green's representation formula.

PDE Notes: 10/1/08 4pages

Recall Green's representation formula

$$u(x) = \int_{\partial\Omega} \left(\Phi(y-x) \frac{\partial u(y)}{\partial \nu} - u(y) \frac{\partial \Phi(y-x)}{\partial \nu} \right) dS(y) - \int_{\Omega} \Phi(y-x) \Delta u(y) \, dy$$

Suppose u solves $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

By Green's representation formula,

$$\Rightarrow u(x) = \int_{\partial\Omega} \left(\Phi(y-x) \underbrace{\frac{\partial u(y)}{\partial \nu}}_{\Downarrow} - g(y) \frac{\partial \Phi(y-x)}{\partial \nu} \right) dS(y) + \int_{\Omega} \Phi(y-x) f(y) \, dy$$

This is unknown.

Need to use the value of u in a nbd of $\partial\Omega$ to compute $\frac{\partial u}{\partial \nu}$. But this info is not given.

So we need to find some way to get rid of $\int_{\partial\Omega} \Phi(y-x) \frac{\partial u(y)}{\partial \nu}$.

Green's ftn:

For fixed $x \in \Omega$, define a corrector ftn $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta_y \phi^x(y) = 0 \\ \phi^x(y) = \Phi(y-x) & \text{when } y \in \partial\Omega \end{cases}$$

Note that $\phi^x(y)|_{\partial\Omega}$ is a "smooth ftn".

The solution $\phi^x(y)$ is smooth.

Apply Green's 2nd Identity

$$\int_\Omega \Big[u(y)\underbrace{\Delta \phi^x(y)}_{=0} - \phi^x(y)\Delta u(y)\Big]dy$$

$$= \int_{\partial\Omega}\Big[u(y)\frac{\partial \phi^x(y)}{\partial \nu} - \underbrace{\phi^x(y)}_{=\Phi(y-x)}\frac{\partial u(y)}{\partial \nu}\Big]dS(y)$$

$$\Rightarrow \quad -\int \phi^x(y)\Delta u(y)\,dy$$

$$② \quad = \int_{\partial\Omega}\Big[u(y)\frac{\partial \phi^x(y)}{\partial \nu} - \underbrace{\Phi(y-x)\frac{\partial u(y)}{\partial \nu}}\Big]dS(y)$$

Recall

$$① \quad u(x) = \int_{\partial\Omega}\Big[\underbrace{\Phi(y-x)\frac{\partial u(y)}{\partial \nu}} - u(y)\frac{\partial \Phi(y-x)}{\partial \nu}\Big]dS(y) - \int_\Omega \Phi(y-x)\Delta u(y)\,dy$$

$$①+② \Rightarrow u(x) = \int_{\partial\Omega} u(y)\Big(\frac{\partial}{\partial \nu}\big(\phi^x(y)-\Phi(y-x)\big)\Big)dS(y)$$

$$+\int_\Omega \big(\phi^x(y)-\Phi(y-x)\big)\Delta u(y)\,dy$$

Def. Green's fn for Ω is

$$G(x,y) = \Phi(y-x) - \underline{\phi^x(y)}$$

defined earlier.

$$\Rightarrow u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) - \int_\Omega G(x,y)\Delta u(y)\,dy$$

Remark

① If $\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

$$\Rightarrow u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu}\,dS(y) + \int_\Omega G(x,y)f(y)\,dy$$

② If u soln $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

$$\Rightarrow u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu}\,dS(y)$$

Th (Symmetry of Green's ftn).

For $x, y \in \Omega$, $x \neq y$,

we have $G(x,y) = G(y,x)$

pf: Fix $x, y \in \Omega$, $x \neq y$.

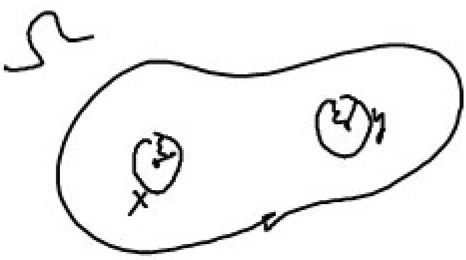
Write $V(z) = G(x,z)$ and $W(z) = G(y,z)$.

Recall that $G(x,z) = \underbrace{\Phi(z-x)}_{\substack{\text{smooth in} \\ \Omega \setminus \{x\}}} - \underbrace{\phi^x(z)}_{\text{smooth}}$

$\Rightarrow$ V is smooth in $\Omega \setminus \{x\}$.

W is smooth in $\Omega \setminus \{y\}$.

$\Rightarrow$ V, W is smooth in $\Omega \setminus \{x, y\}$



Let ε small enough so that

$B(x,\varepsilon) \cap B(y,\varepsilon) = \phi$,

$B(x,\varepsilon) \subset \Omega$, $B(y,\varepsilon) \subset \Omega$

$\Rightarrow$ V and W are smooth in $\Omega \setminus \left(B(x,\varepsilon) \cup B(y,\varepsilon)\right)$

$\Rightarrow$ V and W are smooth in $\Omega \setminus \left(B(x,\varepsilon) \cup B(y,\varepsilon) \right)$

$\Rightarrow$ Apply Green's 2nd identity in

$$0 = \int_{\Omega \setminus (B(x,\varepsilon) \cup B(y,\varepsilon))} \left(V(z) \overset{=0}{\Delta_z W} - W(z) \overset{=0}{\Delta_z V} \right) dz$$

$$= \int_{\partial(\Omega \setminus B(x,\varepsilon) \cup B(y,\varepsilon))} \left(V \frac{\partial W}{\partial \nu} - W \frac{\partial V}{\partial \nu} \right) dS(z)$$

(From def, $\Delta_z V = 0$ in $\Omega \setminus \{x\}$

$\Delta_z W = 0$ in $\Omega \setminus \{y\}$.

$V(z) = \Phi(x,z) - \phi^x(z) = 0$ on $\partial\Omega$

$W(z) = \Phi(y,z) - \phi^y(z) = 0$ on $\partial\Omega$

$$\Rightarrow \int_{\partial B(x,\varepsilon)} \left(V \frac{\partial W}{\partial \nu} - W \frac{\partial V}{\partial \nu} \right) + \int_{\partial B(y,\varepsilon)} \left(V \frac{\partial W}{\partial \nu} - W \frac{\partial V}{\partial \nu} \right) = 0$$

$$\Rightarrow \int_{\partial B(x,\varepsilon)} \left(W \frac{\partial V}{\partial \nu} - V \frac{\partial W}{\partial \nu} \right) dS(y)$$

$$= \int_{\partial B(y,\varepsilon)} \left(V \frac{\partial W}{\partial \nu} - W \frac{\partial V}{\partial \nu} \right) dS(y)$$

Claim: $\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \left(W \frac{\partial V}{\partial \nu} - V \frac{\partial W}{\partial \nu} \right) dS(y)$

$= G(y,x)$ and

$\lim_{\varepsilon \to 0} \int_{\partial B(y,\varepsilon)} \left(V \frac{\partial W}{\partial \nu} - V \frac{\partial V}{\partial \nu} \right) dS(y)$

$= G(x,y)$

$$\Longrightarrow \quad G(y,x) = G(x,y)$$

In the following, we'll construct Green's fth for the unit ball.

Def. $x \in \mathbb{R}^n \setminus \{0\}$, the point $\tilde{x} = \frac{x}{|x|^2}$ is called the point dual to x with respect to $\partial B(0,1)$.

The mapping $x \longrightarrow \tilde{x}$ is inversion thru the unit sphere.

Remark. ① If $\|x\| = R \Rightarrow \|\tilde{x}\| = \frac{1}{R}$

② If $|x| < 1 \Rightarrow |\tilde{x}| > 1$

To construct Green's fth on unit ball

We have to find a corrector $\phi^x = \phi^x(y)$ solving
$$\begin{cases} \Delta_y \phi^x(y) = 0 & \text{in } B(0,1) \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial B(0,1) \end{cases}$$

Lemma: $\|y - \tilde{x}\| = \dfrac{\|x-y\|}{\|x\|}$ where $y \in \partial B(0,1)$ (or $\|y\| = 1$)

pf: $\|y-\tilde{x}\|^2 = \|y - \frac{x}{\|x\|^2}\|^2$

$= \|y\|^2 - 2\frac{y \cdot x}{\|x\|^2} + \frac{\|x\|^2}{\|x\|^4}$

$\underset{\|y\|=1}{=} 1 - 2\frac{y\cdot x}{\|x\|^2} + \frac{1}{\|x\|^2}$

$= \frac{1}{\|x\|^2}\left(1 - 2y\cdot x + \|x\|^2\right)$

$= \frac{1}{\|x\|^2}\left(\|y\|^2 - 2y\cdot x + \|x\|^2\right)$

$= \frac{1}{\|x\|^2}\|y-x\|^2$

$\Rightarrow \|y-\tilde{x}\| = \dfrac{\|y-x\|}{\|x\|}$.

Remark:

② If one considers $\|y - ax\|$ so that

$$\frac{\|y-ax\|}{\|y-x\|} \text{ is indep of } y,$$

then $a = \frac{1}{\|x\|^2}$

①
$$\frac{\|y-\tilde{x}\|}{\|y-x\|} = \frac{1}{\|x\|} \leftarrow \text{indep of } y \text{ for } |y|=1$$

$$\|x\|\,\|y-\tilde{x}\| = \|y-x\|$$

if $x \in B(0,1)$ and $x \neq 0 \Rightarrow \tilde{x} \notin B(0,1)$
$(\tilde{x} \in \mathbb{R}^n \setminus \overline{B(0,1)})$ In fact

Define $\phi^x(y) = \Phi(\|x\|(y-\tilde{x}))$

Recall that if u is harmonic
$\Rightarrow u(cx)$ is also harmonic

$$\left(\Delta u(cx) = c^2 \Delta_y u(\underbrace{cx}_{y}) \right)$$

So $\phi^x(y)$ is harmonic (b/c Φ is harmonic)

$$\Rightarrow \Delta_y \phi^x(y) = 0 \left(\Delta_y \left(\Phi(\|x\|(y-\tilde{x})\right) = \|x\|^2 \Delta_z \overline{\Phi}(z) = 0 \right)$$

Also $\phi^x(y)$

$$= \Phi(\|x\|(y-\tilde{x})) = \Phi(\|x\|\,\|y-\tilde{x}\|)$$
$$= \Phi(\|y-x\|) = \Phi(y-x) \text{ when } y \in \partial B(0,1)$$

So $\phi^x(y) = \Phi\left(\|x\|(y-\tilde{x})\right)$ is smooth.

satisfies

$$\begin{cases} \Delta_y \phi^x(y) = 0 & \text{in } B(0,1) \\ \phi^x(y) = \Phi(y-x) & \text{when } y \in \partial B(0,1) \end{cases}$$

(Recall that $\Phi\left(\|x\|(y-\tilde{x})\right)$ is not defined at $\tilde{x} \notin B(0,1)$)

Def: Green's ftn for the unit ball is

$$G(x,y) := \Phi(y-x) - \Phi\left(\|x\|(y-\tilde{x})\right)$$
$$x, y \in B(0,1)$$

Remark: The def only works for $x \neq 0$

But if $x = 0$ $\Phi(y-x) = \Phi(y)$
$= \text{constant} = C$
when $y \in \partial B(0,1)$

We can choose $\phi^x(y) \equiv \text{constant}$

when $x = 0$.

PDE

Mao-Pei Tsui

University of Toledo

Plan of today's talk

- A overview of the proof of Green's representation formula

- Green's function

- Properties of Green's function

Green's representation formula

Green's representation formula Let $u \in C^2(\overline{\Omega})$.
Then $u(x) =$
$$\int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi(y-x)}{\partial \nu})dS(y) - \int_\Omega \Phi(y-x)\triangle u(y)dy$$

Key points in the proof:

1. Apply Green's second identity to $\Omega \setminus B(x,\epsilon)$ to get
$$\int_{\Omega\setminus B(x,\epsilon)} u(y)\triangle\Phi(y-x) - \Phi(y-x)\triangle u(y)dy =$$
$$\int_{\partial(\Omega\setminus B(x,\epsilon))}(u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u}{\partial \nu}(y))dS(y)$$

2. Use the fact that $\triangle_y\Phi(y-x) = 0$ (regard this as a function of y) to get
$$\int_{\Omega\setminus B(x,\epsilon)} -\Phi(y-x)\triangle u(y)dy =$$
$$\int_{\partial(\Omega\setminus B(x,\epsilon))}(u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u}{\partial \nu}(y))dS(y)$$

3. Note that $\partial(\Omega \setminus B(x,\epsilon) = \partial\Omega \cup \partial B(x,\epsilon)$. So we have
$\int_{\Omega\setminus B(x,\epsilon)} -\Phi(y-x)\triangle u(y)dy$
$= \int_{\partial\Omega}(u(y)\frac{\partial\Phi(y-x)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y)$
$+ \int_{\partial B(x,\epsilon)}(u(y)\frac{\partial\Phi(y-x)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y)$.
Note that the unit normal vector is the exterior normal vector with respect to $\Omega \cup \partial B(x,\epsilon)$.

4. Show that $\int_{\partial B(x,\epsilon)} u(y)\frac{\partial\Phi(y-x)}{\partial\nu}dS(y) = \frac{\int_{\partial B(x,\epsilon)} u(y)dS(y)}{n\alpha_n\epsilon^{n-1}}$ and
$|\int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial\nu}(y)dS(y)| \leq \begin{cases} C\epsilon\ln\varepsilon & n=2 \\ C\epsilon & \text{if } n \geq 3 \end{cases}$
In particular, $\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} u(y)\frac{\partial\Phi(y-x)}{\partial\nu}dS(y) = u(x)$
and $\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial\nu}(y)dS(y) = 0$

5. Now $\lim_{\epsilon\to 0}\int_{\Omega\setminus B(x,\epsilon)} -\Phi(y-x)\triangle u(y)dy$
$= \int_{\partial\Omega}(u(y)\frac{\partial\Phi(y-x)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y)$
$+\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)}(u(y)\frac{\partial\Phi(y-x)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y)$ gives

$\int_{\Omega} -\Phi(y-x)\triangle u(y)dy$
$= \int_{\partial\Omega}(u(y)\frac{\partial\Phi(y-x)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y) + u(x)$ which is equivalent to Green's representation formula.

Recall (Green's representation formula) $u(x) = \int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial\Phi(y-x)}{\partial\nu})dS(y) - \int_{\Omega}\Phi(y-x)\triangle u(y)dy$

Suppose u satisfies $\begin{cases} -\triangle u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

By Green's representation formula, we have $u(x) = \int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial\nu}(y) - g(y)\frac{\partial\Phi(y-x)}{\partial\nu})dS(y) + \int_{\Omega}\Phi(y-x)f(y)dy$.

The drawback of this formula is that it involves the normal derivative of u on the boundary (which is unknown).

So we must find someway to remove $\int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial\nu}(y)dS(y)$.

For fixed $x \in \Omega$, define a corrector function $\phi^x = \phi^x(y)$ a function of y solving $\begin{cases} \triangle_y \phi^x(y) = 0 & \text{in } \Omega \\ \phi^x(y) = \Phi(y-x) & \text{when } y \in \partial\Omega \end{cases}$

Note that $\Phi(y-x)$ is a smooth function in y when $y \in \partial\Omega$ and $x \in \Omega$.

By Green's second identity

$$\int_\Omega u(y)\triangle\phi^x(y) - \phi^x(y)\triangle u(y)dy$$

$$= \int_{\partial\Omega} (u(y)\frac{\partial\phi^x(y)}{\partial\nu} - \phi^x(y)\frac{\partial u}{\partial\nu}(y))dS(y)$$

Using $\triangle_y\phi^x(y) = 0$ and $\phi^x(y) = \Phi(y-x) text\ when\ y \in \partial\Omega$, we have

$$-\int_\Omega \phi^x(y)\triangle u(y)dy$$

$$= \int_{\partial\Omega} (u(y)\frac{\partial\phi^x(y)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y)$$

Combining $u(x) = \int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial \Phi(y-x)}{\partial\nu})dS(y) - \int_\Omega \Phi(y-x)\Delta u(y)dy$ and

$$0 = \int_{\partial\Omega}(u(y)\frac{\partial \phi^x(y)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y) + \int_\Omega \phi^x(y)\triangle u(y)dy,$$

we have $u(x) = \int_{\partial\Omega}(u(y)\frac{\partial(\phi^x(y)-\Phi(y-x))}{\partial\nu})dS(y) + \int_\Omega(\phi^x(y) - \Phi(y-x))\triangle u(y)dy$

Definition: Green's function for the region Ω is

$$G(x,y) = \Phi(y-x) - \phi^x(y), \quad (x, y \in \Omega, x \neq y)$$

where $\phi^x = \phi^x(y)$ solves $\begin{cases} \triangle_y \phi^x(y) = 0 & \text{in } \Omega \\ \phi^x(y) = \Phi(y-x) & \text{when } y \in \partial\Omega \end{cases}$

Note that $\triangle_y G(x,y) = \triangle(\Phi(y-z) - \phi^x(y)) = 0$ if $y \neq x$ and $G(x,y) = 0$ if $y \in \partial\Omega$.

Thus we have proved the following theorem.

Theorem (Representation formula using Green's function$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y))dS(y) - \int_{\Omega} G(x,y)\Delta u(y)dy$

Remark:

1. If $-\Delta u = f$ and $u|_{\partial\Omega} = g$ then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y))dS(y) + \int_{\Omega} G(x,y)f(y)dy.$$

2. If $\Delta u = 0$ and $u|_{\partial\Omega} = g$ then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y))dS(y).$$

Now we want to look at an important property of Green's function.
Theorem (Symmetry of Green's function For all $x, y \in \Omega$, $x \neq y$, we have

$$G(x,y) = G(y,x)$$

Proof: Fix $x, y \in \Omega$, $x \neq y$.
Write $v(z) = G(x,z)$ and $w(z) = G(y,z)$. Then v and w are smooth and harmonic in $\Omega \setminus \{x,y\}$.
Note that $\triangle_z v = \triangle_z w = 0$ and $v = w = 0$ on $\partial\Omega$.
Choose $\epsilon > 0$ small enough so that $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$, $B(x,\epsilon) \subset \Omega$ and $B(y,\epsilon) \subset \Omega$.
Now use Green's second identity in $\Omega \setminus (B(x,\epsilon) \cup B(y,\epsilon))$. We have $0 = \int_{\Omega\setminus(B(x,\epsilon)\cup B(y,\epsilon))}(v(z)\underbrace{\triangle_z w}_{=0} - w(z)\underbrace{\triangle_z v}_{=0})dz$

$= \int_{\partial(\Omega\setminus(B(x,\epsilon)\cup B(y,\epsilon)))}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z)$
$= \int_{\partial\Omega} \underbrace{v(z)}_{=0}\frac{\partial w}{\partial \nu}(z) - \underbrace{w(z)}_{=0}\frac{\partial v}{\partial \nu}(z))dS(z)$
$+ \int_{\partial(B(x,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z)$
$+ \int_{\partial(B(y,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z)$

Thus we have $\int_{\partial B(x,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z)$
$+\int_{\partial B(y,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z) = 0.$

This implies that $\int_{\partial B(x,\epsilon)}(w(z)\frac{\partial v}{\partial \nu}(z) - v(z)\frac{\partial w}{\partial \nu}(z))dS(z)$
$= \int_{\partial B(y,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z)$. Recall that v is smooth at $\Omega \setminus \{x\}$ and w is smooth at $\Omega \setminus \{y\}$.

$$\begin{aligned}
&\int_{\partial B(x,\epsilon)}(w(z)\frac{\partial v}{\partial \nu}(z) - v(z)\frac{\partial w}{\partial \nu}(z))dS(z) \\
&= \int_{\partial B(x,\epsilon)}(G(y,z)\frac{\partial G(x,z)}{\partial \nu} - \underbrace{G(x,z)}\frac{\partial G(y,z)}{\partial \nu})dS(z) \\
&= \int_{\partial B(x,\epsilon)} \underbrace{G(y,z)}_{smooth \quad near \quad x} \frac{\partial}{\partial \nu}(\underbrace{\Phi(z-x)}_{\text{singular when z= x}} - \underbrace{\phi^x(z)}_{smooth \quad near \quad x})dS(z) \\
&- \int_{\partial B(x,\epsilon)}(\underbrace{\Phi(z-x)}_{=C\epsilon^{2-n}} - \underbrace{\phi^y(x)}_{smooth \;\; function})\frac{\partial}{\partial \nu}(\underbrace{\Phi(z-y) - \phi^y(z)}_{smooth \quad near \quad x}))dS(z).
\end{aligned}$$

Note that $Area(\partial B(x,\epsilon)) = C\epsilon^{n-1}$. So we have
$lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)}(w(z)\frac{\partial v}{\partial \nu}(z) - v(z)\frac{\partial w}{\partial \nu}(z))dS(z)$
$= lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} \underbrace{G(y,z)}_{smooth \quad near \quad x} \frac{\partial}{\partial \nu}\Phi(z-x)dS(z)$

Recall that
$\int_{\partial B(x,\epsilon)} u(z)\frac{\partial}{\partial \nu}\Phi(z-x)dS(z) = \frac{1}{n\alpha_n\epsilon^{n-1}}\int_{\partial B(x,\epsilon)} u(z)dS(z)$ where u is smooth in $B(x,\epsilon)$.
(This is proved in the proof of Green's representation formula.)

Hence we have
$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} \underbrace{G(y,z)}_{smooth \quad near \quad x} \frac{\partial}{\partial \nu}\Phi(z-x)dS(z) = G(y,x)$. So
$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)}(w(z)\frac{\partial v}{\partial \nu}(z) - v(z)\frac{\partial w}{\partial \nu}(z))dS(z) = G(y,x)$.
Similarly,
$\lim_{\epsilon\to 0}\int_{\partial B(y,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z) = G(x,y)$.
Thus we have $G(x,y) = G(y,x)$.

PDE NOTES
on Oct 6
6 pages

Recall that:

Last week, we derived the Green's fctn on the unit ball

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$$

where $\tilde{x} = \frac{x}{|x|^2}$

and $\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| \\ \frac{1}{n(n-2)\alpha_n} |x|^{2-n} \end{cases}$

Now we can use the formula of Green's fctn to solve the boundary problem

$$* \begin{cases} \Delta u = 0 & \text{in } B(0,1) \\ u = g & \text{in } \partial B(0,1) \end{cases}$$

Thm: The sol to $*$ is

$$u(x) = \frac{1-|x|^2}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n} \, dS(y)$$

pf: Recall from the representation formula using Green's fn for the sol of $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g & \text{in } \partial\Omega \end{cases}$

$$\Rightarrow u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G(x,y)}{\partial \nu} \, dS(y) + \int_{\Omega} f(y) G(x,y) \, dy$$

Now $\Delta u = 0$ in $B(0,1)$, $u|_{\partial B(0,1)} = g$.

and $G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$

$$\Rightarrow u(x) = -\int_{\partial B(0,1)} g(y) \underbrace{\frac{\partial G(x,y)}{\partial \nu}} \, dS(y)$$

Claim: $\left(\frac{\partial G(x,y)}{\partial \nu} = \nabla_y G(x,y) \cdot \nu \right)$

$$\nabla_y G(x,y) = \frac{(|x|^2 - 1)}{n\alpha_n |x-y|^n} y$$

$\nu = y$ when $y \in \partial B(0,1)$

In $B(0,1)$, $\nu = y$ on $\partial B(0,1)$

$$\Rightarrow \frac{\partial G(x,y)}{\partial \nu} = \nabla_y G(x,y) \cdot \nu = \frac{(|x|^2-1)}{n\alpha_n |x-y|^n} y \cdot y = \frac{|x|^2-1}{n\alpha_n |x-y|^n} \quad \text{when } y \in \partial B(0,1) \; (|y|^2 = 1)$$

$$\Rightarrow u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G(x,y)}{\partial \nu} \, dS(y)$$

$$= -\int_{\partial B(0,1)} g(\underline{y}) \left(\frac{|x|^2 - 1}{n\alpha_n |x-y|^n} \right) dS(\underline{y})$$

$$= \frac{1 - |x|^2}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n} \, dS(y)$$

pf of the claim

$$\nabla_y G(x,y) = \frac{(|x|^2-1)}{n\alpha_n |x-y|^n} y \quad \text{when } |y|=1$$
$$(y \in \partial B(0,1))$$

Recall $G(x,y) = \underbrace{\Phi(y-x)} - \underbrace{\Phi(|x|(y-\tilde{x}))}$

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| & n=2 \\ \frac{1}{n(n-2)\alpha_n} |x|^{2-n} & n \geq 3 \end{cases}$$

In the following, we assume $(n \geq 3)$

$$\frac{\partial \Phi(y-x)}{\partial y_i} = \frac{1}{n(n-2)\alpha_n}(2-n)|y-x|^{1-n} \cdot \frac{\partial}{\partial y_i}|y-x|$$

$$= \frac{-1}{n\alpha_n}|y-x|^{1-n} \cdot \frac{y_i - x_i}{|y-x|}$$

$$= \frac{x_i - y_i}{n\alpha_n |y-x|^n}$$

$$(f(cy))' = c f'(cy)$$

$$\frac{\partial}{\partial y_i} \Phi(\underbrace{|x|(y-\tilde{x})}_{z}) = |x| \frac{\partial \Phi}{\partial z_i}\Big|_{|x|(y-\tilde{x})}$$

$$= |x| \frac{|x|(\tilde{x}_i - y_i)}{n\alpha_n \left||x|(y-\tilde{x})\right|^n}$$

Recall that $\bar{x} = \frac{x}{|x|^2}$

and $\big| |x| (y - \bar{x}) \big| = |y - x|$ when $|y| = 1$

$$\Rightarrow \frac{\partial}{\partial y_i} \Phi\left(|x|(y-\bar{x})\right) = \frac{|x|^2\left(\frac{x_i}{|x|^2} - y_i\right)}{n\alpha_n |x-y|^n}$$

$$= \frac{(x_i - |x|^2 y_i)}{n\alpha_n |x-y|^n} \quad \text{when } |y| = 1$$

$$\Rightarrow \frac{\partial G(x,y)}{\partial y_i} = \frac{\partial \Phi(y-x)}{\partial y_i} - \frac{\partial}{\partial y_i}\Phi\left(|x|(y-\bar{x})\right)$$

$$= \frac{x_i - y_i - (x_i - |x|^2 y_i)}{n\alpha_n |x-y|^n}$$

$$= \frac{(|x|^2 - 1)\, y_i}{n\alpha_n |x-y|^n}$$

$$\Rightarrow \nabla_y G(x,y) = \frac{(|x|^2-1)\, y}{n\alpha_n |x-y|^n} \quad \text{when } |y| = 1$$

#.

Thm: $$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, dS(y)$$

where u solves

$$\begin{cases} \Delta u = 0 & \text{in } B(0,r) \\ u|_{\partial B(0,r)} = g \end{cases}$$

pf: Let $w(x) = u(rx)$.

Then $\begin{cases} \Delta w = 0 & \text{in } B(0,1) \\ w(y) = g(ry) & \text{when } y \in \partial B(0,1) \end{cases}$

By previous Th,

$$\Rightarrow \underset{\substack{\| \\ u(rx)}}{w(x)} = \frac{1-|x|^2}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(ry)}{|x-y|^n} \, dS(y)$$

Let $rx = z \Rightarrow x = \frac{z}{r}$

$$\Rightarrow u(z) = \frac{1 - \frac{|z|^2}{r^2}}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(ry)}{\left|\frac{z}{r} - y\right|^n} \, dS(y)$$

$$= \frac{r^2 - |z|^2}{n\alpha_n \, r^2} \int_{\partial B(0,1)} \frac{g(ry) \, r^n}{|z - ry|^n} \, dS(y)$$

Let $\bar{y} = ry \Rightarrow dS(\bar{y}) = r^{n-1} dS(y)$

$$\Rightarrow u(z) = \frac{r^2 - |z|^2}{n\alpha_n} \cdot \frac{r^n}{r^2 \cdot r^{n-1}} \int_{\partial B(0,r)} \frac{g(\bar{y})}{|z - \bar{y}|^n} \, dS(\bar{y})$$

$$= \frac{r^2 - |z|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(\bar{y})}{|z - \bar{y}|^n} \, dS(\bar{y})$$

#.

Def: The ftn $K(x,y) = \dfrac{r^2-|x|^2}{n\alpha_n r\,|x-y|^n}$

is called the Poisson Kernel

for the ball $B(0,r)$.

$\left(x \in B(0,r),\ y \in \partial B(0,r) \right)$

Remark: If $x=0$

$$\Rightarrow \quad u(0) = \frac{r^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{\underbrace{|y|^n}_{=r^n}}\, dS(y)$$

$$= \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} g(y)\, dS(y)$$

$\left(u = g \text{ on } \partial B(0,r) \right)$.

$$= \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} u(y)\, dS(y)$$

$\Rightarrow$ We get the mean-value equality

② In the formula above,

We assume $u \in C^2\left(\overline{B(0,r)}\right)$.

PDE NOTES
on Oct 8
6 pages

Recall that we derive the formula

(*)
$$u(x) = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \, dS(y)$$

by assuming that we have a sol to the boundary value problem

$$\begin{cases} \Delta u = 0 \text{ in } B(0,r) \\ u\big|_{\partial B(0,r)} = g \end{cases}$$

We'll show that the formula indeed solves the BVP.

Remark ① Recall $G(x,y) = \Phi(y-x) - \phi^x(y)$

where $\begin{cases} \Delta_y \phi^x(y) = 0 & \text{in } \Omega \\ \phi^x(y) = \Phi(y-x) & \text{when } y \in \partial\Omega \end{cases}$

$\Rightarrow \Delta_y G(x,y) = 0$

Since $G(x,y) = G(y,x)$

$\Rightarrow \Delta_x G(x,y) = 0$

$G(x,y)$ is harmonic in x and y.
when $x \neq y$

② The sol of $\begin{cases} \Delta u = 0 & \text{in } B(0,r) \\ u|_{\partial B(0,r)} = 1 \end{cases}$

is $u = 1$

$\Rightarrow$ By Repsentation formula *

$$\Rightarrow 1 = \frac{r^2 - |x|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{1}{|x-y|^n} \, dS(y)$$

for $x \in B(0,r)$.

Thm: Assume $g \in C^0(\partial B(0,r))$ and define u by

$$u(x) = \frac{r^2 - |x|^2}{n \alpha_n r} \int_{\partial B(0,r)} \frac{g(y)\, dS(y)}{|x-y|^n}.$$

for $x \in B(0,r)$

Then (1) $u \in C^\infty(B(0,r))$

(2) $\Delta u = 0$ in $B(0,r)$

(3) $\lim\limits_{x \to x_0} u(x) = g(x_0)$

where $x_0 \in \partial B(0,r)$

pf: 1° b/c

$$\underbrace{\frac{r^2 - |x|^2}{n \alpha_n r}}_{\text{smooth in } x} \underbrace{\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y)}_{\text{smooth in } x \text{ when } x \in B(0,r)}$$

$$\left(\frac{\partial}{\partial x_i} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, dS(y) = \int_{\partial B(0,r)} \frac{\partial}{\partial x_i} \left(\frac{g(y)}{|x-y|^n} \right) dS(y) \right)$$

$\Rightarrow$ u is smooth in $B(0,r)$

2° Recall $u(x) = -\int_{\partial B(0,r)} g(y) \frac{\partial G(x,y)}{\partial \nu} dS(y)$

$$= -\int g(y) \left(\sum_{i=1}^{n} \frac{\partial G(x,y)}{\partial y_i} \cdot \frac{y_i}{r} \right) dS(y)$$

$$\left(\because \frac{\partial G}{\partial \nu} = \nabla_y G(x,y) \cdot \underset{(11)}{\nu} = \nabla_y G(x,y) \cdot \frac{y}{r} \right.$$

$$\Delta_x u(x) = -\int g(y) \left(\sum_{i=1}^{n} \frac{\partial}{\partial y_i} (\underset{\overset{\parallel}{0}}{\Delta_x G(x,y)}) \cdot \frac{y_i}{r} \right) dS(y)$$

$$= 0 \quad \text{when } x \in B(0,r)$$

3° Recall that

$$1 = \frac{r^2 - |x|^2}{n \alpha_n r} \int_{\partial B(0,r)} \frac{1}{|x-y|^n} dS(y)$$

(This implies that if $x \to \partial B(0,r)$

$\Rightarrow r^2 - |x|^2 \to 0$

$\Rightarrow \int_{\partial B(0,r)} \frac{1}{|x-y|^n} dS(y) \to \infty$)

Fix $x_0 \in \partial B(0,r)$.

$$\Rightarrow g(x_0) = \frac{r^2 - |x|^2}{n \alpha_n r} \int_{\partial B(0,r)} \frac{g(x_0)}{|x-y|^n} dS(y)$$

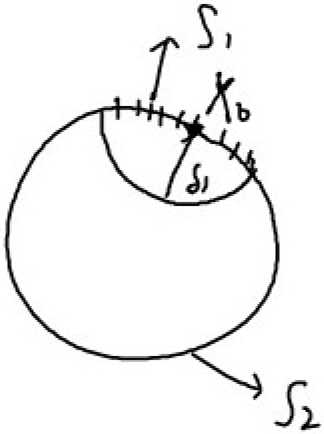
$$\Rightarrow u(x) - g(x_0) = \frac{r^2 - |x|^2}{n \alpha_n r} \int_{\partial B(0,r)} \frac{g(y) - g(x_0)}{|x-y|^n} dS(y)$$

$$u(x)-g(x_0)=\frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)}\frac{g(y)-g(x_0)}{|x-y|^n}\,dS(y)$$

b/c g is cts

We can find $\delta_1>0$

s.t if $|y-x_0|<\delta_1$



$$\Rightarrow \quad |g(y)-g(x_0)|<\frac{\varepsilon}{2}$$

Let $S_1=\{y \mid y\in\partial B(0,r),\ |y-x_0|<\delta_1\}$

$S_2=\{y \mid y\in\partial B(0,r),\ |y-x_0|\geq\delta_1\}$.

$$\Rightarrow |u(x)-g(x_0)|=\frac{r^2-|x|^2}{n\alpha_n r}\left(\int_{S_1}\frac{g(y)-g(x_0)}{|x-y|^n}+\int_{S_2}\frac{g(y)-g(x_0)}{|x-y|^n}\right)$$

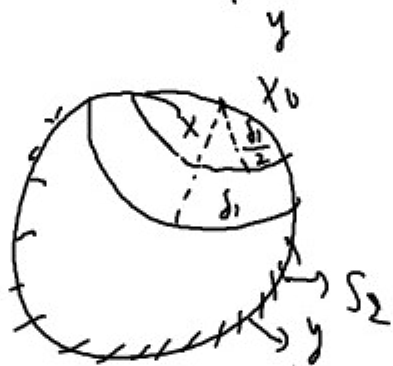
$$\leq\frac{\varepsilon}{2}\cdot\left(\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_1}\frac{1}{|x-y|^n}\right)+\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_2}\frac{|g(y)-g(x_0)|}{|x-y|^n}$$

$$\leq\frac{\varepsilon}{2}\left(\underbrace{\frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)}\frac{1}{|x-y|^n}}_{=1}\right)+$$

$$\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_2}\frac{|g(y)-g(x_0)|}{|x-y|^n}$$

$$\leq\frac{\varepsilon}{2}+2M\cdot\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_2}\frac{1}{|x-y|^n}$$

$(g$ is cts $\Rightarrow |g(y)|\leq M,\ y\in\partial B(0,r))$



Suppose $|x_0 - x| < \frac{\delta_1}{2} \Rightarrow |x-y| \geq \frac{\delta_1}{2}$

$$\Rightarrow \frac{1}{|x-y|^n} \leq \left(\frac{2}{\delta_1}\right)^n$$

$$\Rightarrow 2M \cdot \frac{r^2 - |x|^2}{n\alpha_n r} \int_{S_2} \frac{1}{|x-y|^n}$$

$$\leq 2M \cdot \left(\frac{2}{\delta_1}\right)^n \cdot \left(n\alpha_n r^{n-1}\right) \cdot \frac{r^2 - |x|^2}{n\alpha_n r}$$

if $|x - x_0| < \frac{\delta_1}{2}$

Choose δ_2 s.t. if $|x - x_0| < \delta_2$ $(|x_0| = r)$

$$\Rightarrow \left(2M \cdot \left(\frac{2}{\delta_1}\right)^n (n\alpha_n r^{n-1}) \frac{r^2 - |x|^2}{n\alpha_n r}\right) < \frac{\varepsilon}{2}$$

Choose $\delta = \min(\delta_2, \delta_1)$

$$\Rightarrow |x - x_0| < \delta \Rightarrow |u(x) - g(x_0)| < \varepsilon$$

$$\Rightarrow \lim_{x \to x_0} u(x) = g(x_0)$$

#

PDE NOTES
on Oct. 10
6 pages

HW 4(a) should be

$$\Delta(w\phi^4) + 2\nabla V\cdot\nabla(w\phi^4)$$

$$= 2\phi^4 \|\mathrm{Hess}(V)\|^2 + 4\sum_{i,j} \frac{\partial \phi^4}{\partial x_i}\frac{\partial V}{\partial x_j}\frac{\partial^2 V}{\partial x_i \partial x_j}$$

$$+ w\Delta\phi^4 + 2w\nabla V\cdot\nabla\phi^4$$

(4c) $\displaystyle\sup_{B_r} \frac{|\nabla u|}{u} \le \frac{C}{r}$

Last time, $\delta = \min\left(\frac{\delta_1}{2}, \delta_2\right)$

$\neq$ Not δ_1

Dirichlet Principle:

"Least action principle":

The eqs in "physics" or "math", should be the minimizer of certain energy (functional).

Consider the energy functional

$$I(w) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - wf \right) dx$$

where w belongs to $(f \in C^0(\bar{\Omega}))$

the admissable set

$$A = \{ w \in C^2(\bar{\Omega}) \mid w = g \text{ on } \partial\Omega \}$$

Th: Assume $u \in C^2(\bar{\Omega})$

and
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases} \quad (**)$$

Then $I(u) = \min_{w \in A} I(w) \leftarrow (*)$

Conversely if $u \in A$ satisfies $*$ then u solves the BVP $**$.

Remark: u is a minimizer of the functional I in A

$\Leftrightarrow$ u solves BVP in A.

pf: Prove $I(u) = \min_{w \in A} I(w)$

$$\int_\Omega (\underbrace{\Delta u + f}_{0})(u-w) = 0$$

$$\underbrace{\int (\Delta u)(u-w)}_{\parallel} + \int f(u-w) = 0$$

$$\int \operatorname{div}\big(\nabla u\,(u-w)\big) - \nabla u \cdot (\nabla u - \nabla w)$$

By assumpt $u = w = g$ on $\partial\Omega$

$\Rightarrow$ $u - w = 0$ on $\partial\Omega$

$$\Rightarrow \int_\Omega \operatorname{div}\big(\nabla u(u-w)\big) = 0$$

$$\int -|\nabla u|^2 + \nabla u \cdot \nabla w + \int f(u-w) = 0$$

$$\Rightarrow \int |\nabla u|^2 - fu + fw = \int \nabla u \cdot \nabla w$$

$$\leq \int \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2$$

$$\Rightarrow \int \left(\frac{1}{2}|\nabla u|^2 - fu\right) \leq \int \left(\frac{1}{2}|\nabla w|^2 - fw\right)$$

$\Rightarrow$ $I(u) \leq I(w)$ for all $w \in A$

$$\Rightarrow I(u) = \min_{w \in A} I(w)$$

Now we want to prove that if $I(u) = \min_{w \in A} I(w)$

$\Rightarrow$ u solves $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

Fix $\phi \in C_0^\infty(\Omega)$ $\left(\Rightarrow \phi = 0 \text{ near } \partial\Omega \right)$

Consid $u + \tau\phi$ where $\tau \in \mathbb{R}$

$\Rightarrow$ $u + \tau\phi|_{\partial\Omega} = u|_{\partial\Omega} = g$

Also $u + \tau\phi \in C^2(\bar{\Omega})$

$\Rightarrow$ $u + \tau\phi \in A$

Let $F(\tau) = I(u + \tau\phi)$, $\tau \in \mathbb{R}$

$\Rightarrow$ $F(0) = I(u)$ and achives its minimum at $\tau = 0$

Also F is smooth τ $\Rightarrow F'(0) = 0$

Recall $I(w) = \int \frac{1}{2} |\nabla w|^2 - wf$

$\Rightarrow$ $F(\tau) = I(u + \tau\phi)$

$= \int \frac{1}{2} |\nabla u + \tau \nabla \phi|^2 - (u + \tau\phi) f$

$$= \int \frac{1}{2}\left(|\nabla u|^2 + \tau^2|\nabla\phi|^2 + 2\tau\,\nabla u\cdot\nabla\phi\right)$$

$$- \int (u+\tau\phi)f$$

indep of τ

$$\Rightarrow F'(\tau) = \int \tau|\nabla\phi|^2 + \nabla u\cdot\nabla\phi - \phi f$$

$$F'(0) = \int_\Omega \nabla u\cdot\nabla\phi - \phi f$$

$$F'(0)=0 \iff \int_\Omega \nabla u\cdot\nabla\phi - \phi f = 0$$

$$\Rightarrow \int_\Omega \underbrace{\operatorname{div}(\phi\nabla u)}_{=0} - \phi\Delta u - \phi f = 0$$

(b/c $\phi \in C_0^\infty(\Omega)$, $\phi = 0$ on $\partial\Omega$)

$$\Rightarrow \int_\Omega \phi\,(\underbrace{\Delta u + f}_{\text{cts ftn}}) = 0 \quad \text{for any } \phi \in C_0^\infty(\Omega)$$

$$\Rightarrow \Delta u + f = 0 \text{ in } \Omega$$

$$\Rightarrow -\Delta u = f$$

b/c $u \in A \Rightarrow u|_{\partial\Omega} = g$

$\Rightarrow$ u solves BVP.

Remark: u solves BVP

$$\Rightarrow \underbrace{\int_\Omega \nabla u\cdot\nabla\phi}_{\text{defined when } u\in C^1(\Omega)} = \int_\Omega \phi f$$

$\langle u,\phi\rangle = \int_\Omega \nabla u\cdot\nabla\phi$ is a inner product

$\phi \longmapsto \int \phi f$ a linear functional

Consider the completion of $C_0^\infty(\Omega)$ w.r.t. this inner product

Sobolev Sp

PDE NOTES
on Oct. 13
6 pages

Ch 5 (Trans)

Sobolev Spaces

(Integration by parts formula)

Let $u, \phi \in C^1(\bar{\Omega})$

Then $$\int_\Omega \frac{\partial u}{\partial x_i} \phi \, dx = -\int_\Omega u \frac{\partial \phi}{\partial x_i} dx + \int_{\partial\Omega} u \phi \nu^i dS$$

where $\nu = (\nu^1, \nu^2, \cdots, \nu^n)$ is the outward unit normal vector on $\partial\Omega$

pf. $$\int_\Omega \operatorname{div}\left((0, 0, \cdots, \underset{i\text{th component}}{u\phi}, 0 \cdots 0)\right)$$

$$= \int_{\partial\Omega} \underbrace{(0, \cdots 0, u\phi, 0 \cdots 0) \cdot (\nu_1, \cdots, \nu_i, 0 \cdots 0)}_{= u\phi\nu_i} dS(x)$$

$$\operatorname{div}(0, \cdots 0, u\phi, 0 \cdots 0) = \frac{\partial}{\partial x_i}(u\phi)$$

$$= \frac{\partial u}{\partial x_i}\phi + u\frac{\partial \phi}{\partial x_i}$$

$$\Rightarrow \int_\Omega \left(\frac{\partial u}{\partial x_i}\phi + u\frac{\partial \phi}{\partial x_i}\right) = \int_{\partial\Omega} u\phi\nu_i \, dS(x)$$

$$\Rightarrow \int_\Omega \frac{\partial u}{\partial x_i}\phi = -\int_\Omega u\frac{\partial \phi}{\partial x_i} + \int_{\partial\Omega} u\phi\nu_i \, dS(x)$$

Cor: $\int_\Omega \frac{\partial u}{\partial x_i} \phi = - \int_\Omega u \frac{\partial \phi}{\partial x_i}$ if $\phi \in C_0^\infty(\Omega)$

pf: b/c $\phi = 0$ on $\partial\Omega$

Cor: $\int_\Omega D^\alpha u \, \phi = (-1)^{|\alpha|} \int_\Omega u \, D^\alpha \phi$ — (*)

if $u \in C^k(\bar{\Omega})$, $|\alpha| \le k$
$\phi \in C_0^\infty(\Omega)$

Recall: $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_n)$
$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

$C_0^\infty(\Omega) = \{ \phi$ is smooth in Ω with compact support $\}$

RHS of * makes sense as long as $u D^\alpha \phi$ is integrable.

This leads to the following definition

Def. Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multi-index, we say v is the α-th weak partial derivative of u, written $D^\alpha u = v$

if $\int_\Omega u \, D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v \phi \, dx$

for all test ftns $\phi \in C_0^\infty(\Omega)$.

Lemma: A weak α-th partial derivative of u (if it exists) is uniquely defined up to a set of measure zero.

pf: Suppose $v, \bar{v} \in L^1_{loc}(\Omega)$ are both α-th weak partial derivative of u

$$\Rightarrow \int_\Omega u\, D^\alpha \phi\, dx = (-1)^{|\alpha|}\int_\Omega v\phi = (-1)^{|\alpha|}\int_\Omega \bar{v}\phi$$

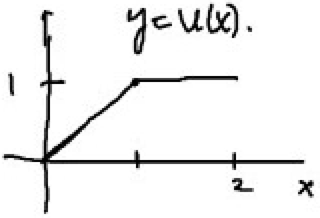
$$\Rightarrow \int_\Omega (v-\bar{v})\phi = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega)$$

$$\Rightarrow v - \bar{v} = 0 \text{ a.e.}$$

Remark: The weak partial derivative of a fn is unique (in $L^1_{loc}(\Omega)$).

Ex. Let $n=1$, $\Omega = (0,2)$.

$$u(x) = \begin{cases} x & 0 < x \le 1 \\ 1 & 1 \le x < 2 \end{cases}$$



$u \notin C^1(\Omega)$

Define $v = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 < x < 2 \end{cases}$

We want to show that v

is the weak 1-st derivative of u

We have to show that

$$\int_0^2 u\phi' \, dx = (-1)\int_0^2 v\phi \, dx \text{ for } \phi \in C_0^\infty(\Omega)$$

$$\int_0^2 u\phi' \, dx = \int_0^1 x\phi' dx + \int_1^2 \phi' dx$$

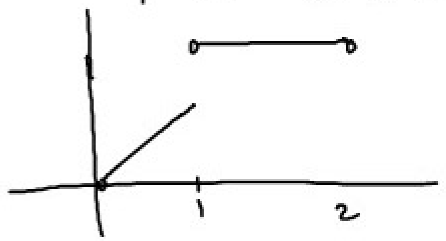
$$= x\phi\Big]_0^1 - \int_0^1 \phi \, dx + \phi(2) - \phi(1)$$

$$= \underbrace{\phi(1)} - 0 - \int_0^1 \phi \, dx + \phi(2) - \underbrace{\phi(1)}$$

$$= -\int_0^1 \phi \, dx \quad \left(\text{b/c } \phi \in C_0^\infty((0,2)) \Rightarrow \phi(2) = 0 \right)$$

$$= -\int_0^2 v \phi \, dx \quad \left(\text{b/c } v = \begin{cases} 1 & 0 < x \le 1 \\ 0 & 1 < x < 2. \end{cases} \right.$$

Ex: $u(x) = \begin{cases} x & 0 < x \le 1 \\ 2 & 1 < x < 2 \end{cases}$



The weak derivative of u

doesn't exist. (see example 2 on p244).

Definition of Sobolev Spaces.

① $W^k(\Omega) = \left\{ u \in L^1_{loc}(\Omega) \,\middle|\, \begin{array}{l} \text{the weak derivative} \\ D^\alpha u \text{ exists for} \\ |\alpha| \le k \end{array} \right\}$

Remark. $C^k(\Omega) \subset W^k(\Omega)$

② $W^{k,p}(\Omega) = \left\{ u \in W^k(\Omega) \,\middle|\, \begin{array}{l} D^\alpha u \in L^p(\Omega) \\ \text{for all } |\alpha| \le k \end{array} \right\}$

($|\alpha| = 0$ means $u \in L^p(\Omega)$)

$W^{k,p}(\Omega) \subset L^p(\Omega)$

Def. If $u \in W^{k,p}(\Omega)$, we define the norm

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}$$

$$\left(= \sum_{|\alpha| \le k} \left(\int |D^\alpha u|^p dx \right)^{\frac{1}{p}} \right.$$

$1 \le p < \infty$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \le k} \operatorname{ess\,sup}_{\Omega} |D^\alpha u|$$

Remark. We'll show that

$$\left(W^{k,p}, \| \ \|_{W^{k,p}(\Omega)}\right)$$

is a Banach space later

Def. ① Let $\{u_m\}_{m=1}^{\infty}$, $u \in W^{k,p}(\Omega)$

We say u_m converges to u in $W^{k,p}(\Omega)$, write $u_m \longrightarrow u$ in $W^{k,p}(\Omega)$

if $\lim\limits_{m\to\infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$

② $u_m \longrightarrow u$ in $W^{k,p}_{loc}(\Omega)$

if $u_m \longrightarrow u$ in $W^{k,p}(V)$ for each $V \subset\subset \Omega$.

Def. $W_0^{k,p}(\Omega) =$ the closure of $C_0^\infty(\Omega)$ in $\left(W^{k,p}(\Omega), \| \ \|_{W^{k,p}}\right)$.

$u \in W_0^{k,p}(\Omega) \Longleftrightarrow \exists\ u_m \in C_0^\infty(\Omega)$

st $\lim\limits_{m\to\infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$

$$H_0^k(\Omega) = W_0^{k,2}(\Omega) \subset L^2(\Omega)$$

We'll show that this is a Hilbert space later.

Recall the Sobolev Space

PDE NOTES
on Oct. 15
4 pages

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \text{the weak derivative} \\ D^\alpha u \text{ exists for } |\alpha| \le k \\ D^\alpha u \in L^p(\Omega) \end{array} \right\}.$$

$D^\alpha u$ is the weak partial derivative of u

iff $$\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega (D^\alpha u)\, \phi$$

for $\phi \in C_0^\infty(\Omega)$

In the book, the $W^{k,p}$ norm is defined

by $$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

(When $\alpha = 0$, $\|D^0 u\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega)}$)

One can define another norm

$$\|u\|'_{W^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)}$$

In fact, these two norms are equivalent

Obviously, $\|u\|_{W^{k,p}(\Omega)} \le \|u\|'_{W^{k,p}(\Omega)}$

Recall that $\left|\sum a_i b_i\right| \le \left(\sum_i a_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_i b_i^q\right)^{\frac{1}{q}}$

when $\frac{1}{p}+\frac{1}{q}=1$

$$\Rightarrow \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)} \cdot 1\right) \le \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p}\right)^{\frac{1}{p}} \cdot \underbrace{\left(\sum_{|\alpha|\le k} 1\right)^{\frac{1}{q}}}_{= C_{k,q}}$$

$$\Rightarrow \quad \|u\|'_{W^{k,p}(\Omega)} \le \|u\|_{W^{k,p}(\Omega)} \cdot \underbrace{C_{k,q}}_{>0}$$

$\Rightarrow$ $\|\ \|'_{W^{k,p}}$ and $\|\ \|_{W^{k,p}}$ are equivalent.

Properties of Weak derivatives.

Th Assume $u, v \in W^{k,p}(\Omega)$

Then (1) For each $\lambda, \mu \in \mathbb{R}$

$\Rightarrow \lambda u + \mu v \in W^{k,p}(\Omega)$

(2) Suppose $|\alpha| + |\beta| \le k$.

Then $D^\alpha u \in W^{k-|\alpha|, p}(\Omega)$,

$$D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$$

(3) If $V \underset{\text{open}}{\subset} \Omega \Rightarrow W^{k,p}(\Omega) \subset W^{k,p}(V)$

(4) If $\xi \in C_0^\infty(\Omega)$ then

(a) $\xi \cdot u \in W^{k,p}(\Omega)$

(b) $$D^\alpha(\xi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \xi \cdot D^{\alpha-\beta} u$$

where $\binom{\alpha}{\beta} = \dfrac{\alpha!}{\beta!(\alpha-\beta)!}$

pf: 1° $u, v \in W^{k,p}(\Omega)$, $\phi \in C_0^\infty(\Omega)$

$$\Rightarrow \int_\Omega u\, D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega D^\alpha u\, \phi, \quad |\alpha| \le k$$

and $$\int_\Omega v\, D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega D^\alpha v\, \phi$$

$$\Rightarrow \int_\Omega (\lambda u + \mu v)\, D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega (\lambda D^\alpha u + \mu D^\alpha v)\, \phi$$

for all $\phi \in C_0^\infty(\Omega)$

$$\Rightarrow D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v, \quad |\alpha| \le k$$

Since $D^\alpha u, D^\alpha v \in L^p(\Omega)$ for $|\alpha| \le k$

$\Rightarrow \quad \lambda D^\alpha u + \mu D^\alpha v \in L^p(\Omega)$

$\Rightarrow \quad D^\alpha(\lambda u + \mu v) \in L^p(\Omega)$

$\Rightarrow \quad \lambda u + \mu v \in W^{k,p}(\Omega)$

2° First, Note that

if $\phi \in C_0^\infty(\Omega) \Rightarrow D^\beta \phi \in C_0^\infty(\Omega)$

$|\alpha| + |\beta| \le k$

test ftn

$$\Rightarrow \int u\, D^{\alpha+\beta}\phi = \int u\, D^\alpha(\underline{D^\beta\phi}) = (-1)^{|\alpha|}\int \underbrace{D^\alpha u \cdot (D^\beta \phi)}$$

$$\| $$

$$(-1)^{|\alpha|+|\beta|}\int D^{(\alpha+\beta)} u\, \phi$$

$$\Rightarrow \int_\Omega \underset{\underset{w}{\|}}{(D^\alpha u)} \cdot (D^\beta \phi) = (-1)^{|\beta|}\int_\Omega D^{(\alpha+\beta)} u \cdot \phi$$

$$\Rightarrow \int_\Omega w\, D^\beta \phi = (-1)^{|\beta|}\int_\Omega D^{(\alpha+\beta)} u \cdot \phi \quad \text{for } \phi \in C_0^\infty(\Omega)$$

$\Rightarrow \quad D^\beta w = D^{(\alpha+\beta)} u$

$\Rightarrow \quad D^\beta(D^\alpha u) = D^{(\alpha+\beta)} u$

Similarly $D^\alpha(D^\beta u) = D^{(\alpha+\beta)} u$.

3° If $V \underset{\text{open}}{\subset} \Omega$ and $u \in W^{k,p}(\Omega)$

$$\int_V u\, D^\alpha \phi \qquad \text{when } \phi \in C_0^\infty(V) \subset C_0^\infty(\Omega)$$

$$= \int_\Omega u\, D^\alpha \phi = (-1)^{|\alpha|}\int_\Omega D^\alpha u\, \phi = (-1)^{|\alpha|}\int_V D^\alpha u\, \phi$$

$\Rightarrow \quad u \in W^{k,p}(V) \quad (\; L^p(\Omega) \subset L^p(V))$

We want to prove that

$$D^{\alpha}(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^{\beta}\zeta \cdot D^{\alpha-\beta} u$$

where $\zeta \in C_0^{\infty}(\Omega)$ and $u \in W^{k,p}(\Omega)$.

Prove this by induction on $|\alpha|$.

If $|\alpha| = 1$,

$$\int (\zeta u) D^{\alpha}\phi \underset{\zeta,\phi \in C_0^{\infty}(\Omega)}{=\!=} \int u \underbrace{(\zeta D^{\alpha}\phi)}_{\text{when } |\alpha|=1}$$

$$\underset{\text{product rule}}{=\!=} \int u \left[\underbrace{D^{\alpha}(\zeta\phi) - (D^{\alpha}\zeta)\phi}\right]$$

$$= \int u D^{\alpha}(\zeta\phi) - \int (u D^{\alpha}\zeta)\phi$$

$$\underset{\text{def of weak p.D.}}{\overset{\zeta\phi \in C_0^{\infty}(\Omega)}{=\!=}} (-1)^{|\alpha|} \int (D^{\alpha}u)(\zeta\phi) - \int (u D^{\alpha}\zeta)\phi$$

$$\underset{|\alpha|=1}{=} (-1)^{|\alpha|} \int (D^{\alpha}u\,\zeta + u D^{\alpha}\zeta)\phi$$

$$\Longrightarrow D^{\alpha}(u\zeta) = D^{\alpha}u\,\zeta + u D^{\alpha}\zeta$$

(weak p.D of ζu exists)

So Leibniz's formula holds when $|\alpha| = 1$

Assume the $\mathcal{L}$'s formula holds

when $|\alpha| = K$

Now $|\alpha| = K+1$

We can find β and γ s.t.

$\alpha = \beta + \gamma$ with $|\beta| = K$, $|\gamma| = 1$

Consider $\displaystyle\int (\xi u) D^{\alpha}\phi$

$$= \int (\xi u)\, D^{\beta} \cdot \underbrace{(D^{\gamma}\phi)}_{\substack{\uparrow \\ \text{test fn in } C_0^{\infty}(\Omega)}}$$

$$\underset{\substack{\uparrow \\ \text{By induction} \\ |\beta| = K}}{=} (-1)^{|\beta|} \int D^{\beta}(\xi u) \cdot (D^{\gamma}\phi)$$

$$= (-1)^{|\beta|} \int \left[\sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^{\sigma}\xi \, D^{\beta-\sigma} u \right] D^{\gamma}\phi$$

(By induction the w.p.D of

$D^{\sigma}\xi \cdot D^{\beta-\sigma}u$ exists

$\uparrow$

$C_0^{\infty}(\Omega)$ $\quad |\beta-\sigma| \le K$

Also $W^{k,p}$ is a vector space

$\Rightarrow$ Weak p.D of $\displaystyle\sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^{\sigma}\xi \, D^{\beta-\sigma} u$

exists.

$$= (-1)^{|\gamma| + |\beta|} \int \left[\sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^{\gamma}\left(D^{\sigma}\xi \, D^{\beta-\sigma} u\right) \right] \phi$$

$$= (-1)^{|\alpha|} \int \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left(D^{\gamma+\sigma}\xi \, D^{\beta-\sigma} u + D^{\sigma}\xi \cdot D^{\beta+\gamma-\sigma} u \right) \phi$$

$$= (-1)^{|\alpha|} \int \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \left[\underline{D^{\gamma+\sigma} \xi \, D^{\beta-\sigma} u} + D^{\sigma} \xi \, D^{\gamma+\beta-\sigma} u \right] \phi$$

$$\downarrow$$

$$\underline{\underline{\alpha = \beta + \gamma}} \; (-1)^{|\alpha|} \int \left[\sum_{\rho \le \beta+\gamma} \binom{\beta}{\rho - \gamma} D^{\rho} \xi \, D^{\alpha-\rho} u + \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^{\sigma} \xi \, D^{\alpha-\sigma} u \right] \phi$$

let $\gamma + \sigma = \rho$

$$\begin{cases} \beta - \sigma = (\beta+\gamma) - (\sigma+\gamma) \\ \qquad\; = \alpha - \rho \\ \sigma = \rho - \gamma \end{cases}$$

$$= (-1)^{|\alpha|} \int \left(\sum_{\sigma \le \alpha} \left[\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} \right] D^{\sigma} \xi \, D^{\alpha-\sigma} u \right) \phi$$

$$\| $$

$$= (-1)^{|\alpha|} \int \left(\sum_{\sigma \le \alpha} \binom{\beta+\gamma}{\sigma} D^{\sigma} \xi \, D^{\alpha-\sigma} u \right] \phi$$

$$= (-1)^{|\alpha|} \int \sum_{\sigma \le \alpha} \left(\binom{\alpha}{\sigma} D^{\sigma} \xi \, D^{\alpha-\sigma} u \right) \phi$$

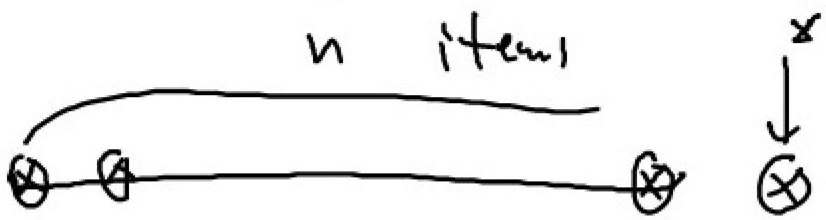
$$\Rightarrow D^{\alpha}(\xi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^{\beta}\xi \, D^{\alpha-\beta} u$$

#.

We have used $\binom{B}{\beta-\gamma} + \binom{B}{\beta} = \binom{B+\gamma}{\beta}$

special case $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

$\binom{n}{k}$ = the # of choice of k items from n items



$\binom{n+1}{k}$ = always pick * + not picking *

$= \binom{n}{k-1} + \binom{n}{k}$

pick k-1 items from the first n items

pick k items from the first n items

We'll prove that

$$\left(W^{k,p}(\Omega), \|\ \|_{W^{k,p}(\Omega)}\right) \text{ or }$$

$$\left(\left(W^{k,p}(\Omega), \|\ \|'_{W^{k,p}(\Omega)}\right)\right)$$

is a Banach space

Recall that A Banach Space is a complete, normed linear space.

Th: The Sobolev space $\left(W^{k,p}(\Omega), \|\ \|_{W^{k,p}(\Omega)}\right)$ or $\left(W^{k,p}(\Omega), \|\ \|'_{W^{k,p}(\Omega)}\right)$ is a Banach Space where k is a positive integer and $1 \le p \le \infty$

pf: 1° From Th I (ii) on p247, We Know $W^{k,p}(\Omega)$ is a linear space.

2° Now, we check that $\|\ \|_{W^{k,p}}$ or $\|\ \|'_{W^{k,p}}$ is a norm.

Recall that X is a Banach space if

(1) X is a linear (vector) space.

(2) X admits a norm, i.e.

$\| \ \| : X \longrightarrow [0, \infty)$ s.t

(a) $\|u\| = 0 \iff u = 0$

(b) $\|\lambda u\| = |\lambda| \, \|u\|$

(c) $\|u+v\| \le \|u\| + \|v\|$

(3) Suppose $\{u_m\}_{m=1}^{\infty}$ is a Cauchy seq

Then we can find $u \in X$ s.t

$$\lim_{m\to\infty} \|u_m - u\| = 0$$

3° Recall $u \in W^{k,p}(\Omega)$

$$\|u\|_{W^{k,p}} = \Big(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}$$

$$\|u\|'_{W^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)}$$

If $\|u\|_{W^{k,p}} = 0 \Rightarrow \sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)} = 0$

Note that $\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p}$

$$= \|u\|^p_{L^p(\Omega)} + \sum_{1\le|\alpha|\le k} \|D^\alpha u\|^p_{L^p}$$

So $\|u\|_{W^{k,p}} = 0 \Rightarrow \|u\|_{L^p(\Omega)} = 0$

$\Rightarrow u = 0$ in $L_p(\Omega)$

Also $u = 0$ is in $W^{k,p}(\Omega)$

$$\|u\|'_{W^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)}$$

$$= \|u\|_{L^p(\Omega)} + \sum_{1\le|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)}$$

$$\|u\|'_{W^{k,p}} = 0 \Rightarrow \|u\|_{L^p(\Omega)} = 0$$

$$\Rightarrow u = 0 \quad \text{in } W^{k,p}(\Omega)$$

4° Now want to show that

$$\|\lambda u\|_{W^{k,p}} = |\lambda|\,\|u\|_{W^{k,p}}$$

and $\|\lambda u\|'_{W^{k,p}} = |\lambda|\,\|u\|'_{W^{k,p}}$

Since $\|\lambda u\|_{W^{k,p}} = \left(\sum_{|\alpha|\le k} \|D^\alpha(\lambda u)\|^p_{L^p}\right)^{\frac{1}{p}}$

$$= |\lambda|\,\|u\|_{W^{k,p}}$$

Similarly, $\|\lambda u\|'_{W^{k,p}} = \sum_{|\alpha|\le k} \overbrace{\|D^\alpha(\lambda u)\|}^{\|\lambda D^\alpha(u)\|_{L^p}\ =}{}_{L^p(\Omega)}$

$$= |\lambda| \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)} = |\lambda|\,\|u\|'_{W^{k,p}}$$

5° Now want to prove that

$$\|u+v\|_{W^{k,p}} \le \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}} \quad \text{and}$$

$$\|u+v\|'_{W^{k,p}} \le \|u\|'_{W^{k,p}} + \|v\|'_{W^{k,p}}$$

Recall Minkowski's inequality

(i) $\|u+v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}$

(ii) $\left(\sum_i (a_i+b_i)^p\right)^{\frac{1}{p}} \le \left(\sum_i a_i^p\right)^{\frac{1}{p}} + \left(\sum_i b_i^p\right)^{\frac{1}{p}}$

$$\|u+v\|'_{W^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha(u+v)\|_{L^p(\Omega)}$$

$$= \sum_{|\alpha|\le k} \|\underbrace{D^\alpha u}_{L^p} + \underbrace{D^\alpha v}_{L^p}\|_{L^p(\Omega)}$$

$$\underset{\text{M ineq.}}{\le} \sum_{|\alpha|\le k} \left(\|D^\alpha u\|_{L^p(\Omega)} + \|D^\alpha u\|_{L^p(\Omega)} \right)$$

$$\|$$

$$\|u\|'_{W^{k,p}} + \|v\|'_{W^{k,p}}$$

$$\|u+v\|_{W^{k,p}} = \left(\sum_{|\alpha|\le k} \|D^\alpha(u+v)\|^p_{L^p} \right)^{\frac{1}{p}}$$

$$\le \left(\sum_{|\alpha|\le k} \left(\|D^\alpha(u)\|_{L^p} + \|D^\alpha v\|_{L^p} \right)^p \right)^{\frac{1}{p}}$$

$$\underset{\substack{\text{Discrete} \\ \text{M-ineq.}}}{\le} \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p \right)^{\frac{1}{p}} + \left(\sum_{|\alpha|\le k} \|D^\alpha v\|^p \right)^{\frac{1}{p}}$$

$$\|$$

$$\|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}$$

6° It remains to show that $W^{k,p}$ is complete.

Since $\| \ \|'$ and $\| \ \|'$ are equivalent, it suffices to show that $(W^{k,p}, \| \ \|')$ is complete.

Assume $\{u_m\}_{m=1}^{\infty}$ is a cauchy seq in $W^{k,p}$ w.r.t $\| \ \|'_{W^{k,p}}$.

Since $\|u_n - u_m\|'_{W^{k,p}}$

$$= \|u_n - u_m\|_{L^p(\Omega)} + \sum_{1\le|\alpha|\le k} \|D^\alpha u_n - D^\alpha u_m\|_{L^p(\Omega)}$$

We know that $\{u_n\}_{n=1}^{\infty}$ is a cauchy seq in L_p and $\{D^\alpha u_n\}_{n=1}^{\infty}$ is also a cauchy seq in L_p.

b/c L_p is a Banach space.

$\Rightarrow$ We can find $u \in L^p$, $u^\alpha \in L^p$ for $|\alpha| \le k$

s.t $\lim_{n\to\infty} \|u_n - u\|_{L^p} = 0$ and

$\lim_{n\to\infty} \|D^\alpha u_n - u^\alpha\|_{L^p} = 0$

$\lim_{n\to\infty} \| D^\alpha u_n - u^\alpha \|_{L^p}$

Now we want to show that $u \in W^{k,p}(\Omega)$ and $D^\alpha u = u^\alpha$

This implies that $\lim_{n\to\infty} \|u_n - u\|_{W^{k,p}} = 0$

$$\left(\lim_{n\to\infty} \|u_n - u\|_{L^p} + \sum_{|\alpha|\le k} \|D^\alpha u_n - \underbrace{D^\alpha u}_{= u^\alpha}\|_{L^p} \right)$$

$\Rightarrow W^{k,p}$ is a Banach space

Given $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega u D^\alpha \phi \, dx = \lim_{m\to\infty} \int u_m D^\alpha \phi \quad \left(\text{b/c } D^\alpha\phi \in L^q(\Omega) \text{ when } \frac{1}{p}+\frac{1}{q}=1 \right)$$

$$\left(\text{b/c } \int |u D^\alpha \phi - u_m D^\alpha \phi| \le \left(\int \underbrace{|u - u_m|}_{L^p} \underbrace{|D^\alpha \phi|}_{L^q} \right) \underset{\text{Holder ineq}}{\le} \left(\int |u - u_m|^p \right)^{\frac{1}{p}} \left(\int (D^\alpha \phi)^q \right)^{\frac{1}{q}} \to 0 \right)$$

$$= \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega D^\alpha u_m \phi \quad \left(\text{b/c } u_m \in W^{k,p} \Rightarrow D^\alpha u_m \text{ exists} \right)$$

$D^\alpha u_m \to u^\alpha$ in L_p

$$= (-1)^{|\alpha|} \int_\Omega u^\alpha \phi$$

$$\Rightarrow \int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega u^\alpha \phi$$

$\Rightarrow D^\alpha u = u^\alpha$

#

PDE NOTES
on Oct.24
6 pages

Ex. Let $\Omega = B(0,1) \subset \mathbb{R}^n$

and $u(x) = |x|^{-\delta}$, $\delta > 0$, $x \in \Omega$, $x \neq 0$

Then ① $u \in L^1(\Omega)$ if $n > \delta$

② $u \in W^1(\Omega)$ if $n > \delta + 1$

③ $u \in W^{1,p}(\Omega)$ iff $\delta < \dfrac{n-p}{p}$

$\Updownarrow$

$p(\delta+1) < n$

pf. Consider $\displaystyle\int_{\Omega \setminus B(0,\varepsilon)} |u(x)|\,dx$

$$= \int_{\Omega \setminus B(0,\varepsilon)} \frac{1}{|x|^\delta}\,\underline{dx}$$

$$= \int_\varepsilon^1 \frac{1}{r^\delta}\,\underline{(n\alpha_n) r^{n-1}\,dr}$$

$$= n\alpha_n \int_\varepsilon^1 r^{n-\delta-1}\,dr$$

$$= n\alpha_n \left.\frac{r^{n-\delta}}{n-\delta}\right]_\varepsilon^1 \quad \text{if } n-\delta \neq 0$$

$$= n\alpha_n \left(\frac{1}{n-\delta} - \frac{\varepsilon^{n-\delta}}{n-\delta}\right)$$

$$\lim_{\varepsilon\to 0} \varepsilon^{n-\delta} = 0 \text{ if } n-\delta > 0 \Longrightarrow \int_{\Omega} |u(x)| < \infty$$

$$\lim_{\varepsilon\to 0} \varepsilon^{n-p} = \infty \text{ if } n-\delta < 0$$

$$\text{if } n-\delta = 0 \Rightarrow \int_{\Omega\setminus B(0,\varepsilon)} |u(x)| = n\alpha_n(-\ln\varepsilon) \xrightarrow{} \infty \text{ as } \varepsilon \to 0$$

$$\Longrightarrow u \in L^1(\Omega) \text{ iff } n-\delta > 0$$

Claim: $u \in W^1(\Omega)$ if $n > \delta + 1$

$$\frac{\partial}{\partial x_i} u = \frac{\partial}{\partial x_i}\left(|x|^{-\delta}\right)$$

$$= -\delta |x|^{-\delta-1} \cdot \frac{x_i}{|x|} \quad \left(\frac{\partial}{\partial x_i}|x| = \frac{x_i}{|x|}\right).$$

$$= -\delta |x|^{-\delta-2} x_i$$

$$\Rightarrow |Du|^2 = \delta^2 |x|^{-2\delta-4} |x|^2 = \delta^2 |x|^{-2\delta-2}$$

$$\Rightarrow |Du| = \frac{\delta}{|x|^{\delta+1}}$$

To show that $u \in W^1(\Omega)$,
we need to prove that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = (-1) \int_\Omega \frac{\partial u}{\partial x_i} \phi$$

Also the weak derivative of u

$$\underbrace{\frac{\partial u}{\partial x}}_{W.D} = \frac{\partial u(x)}{\partial x} \text{ defined when } X \neq 0$$

Consider $\phi \in C_0^\infty(\Omega)$ and for $\varepsilon > 0$

$$\textcircled{A} \int_{\Omega \setminus B(0,\varepsilon)} u \frac{\partial \phi}{\partial x_i} \, dx = -\int_{\Omega \setminus B(0,\varepsilon)} \frac{\partial u}{\partial x_i} \phi \, dx + \int_{\partial B(0,\varepsilon)} u \phi \nu^i \, ds$$

(b/c $\phi = 0$ on $\partial\Omega$)

$$\left| \int_{\partial B(0,\varepsilon)} u \phi \nu^i \, ds \right| \leq \|\phi\|_\infty \int_{\partial B(0,\varepsilon)} |u| \, ds.$$

ν^i = i-th component of the unit normal vectors
$|\nu^i| \leq 1$

$$\int_{\partial B(0,\varepsilon)} |u| \, ds \underset{u = |x|^{-\delta}}{=} \frac{n \alpha_n \varepsilon^{n-1}}{\varepsilon^\delta} = n \alpha_n \varepsilon^{n-1-\delta}$$

If $n-1-\delta > 0$ then $\lim_{\varepsilon \to 0} n \alpha_n \varepsilon^{n-1-\delta} = 0$

$\Rightarrow \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} u \phi \nu^i = 0$, Let $\varepsilon \to 0$ in $*$

$$\Rightarrow \int_\Omega u \frac{\partial \phi}{\partial x_i} = (-1) \int_\Omega \frac{\partial u}{\partial x_i} \phi$$

$\Rightarrow u \in W^1(\Omega)$ if $n-1-\delta > 0$

$u \in W^{1,p}(\Omega)$ iff $\delta < \frac{n-p}{p}$

$D^i u = \frac{\partial u}{\partial x_i}$ if $x \neq 0$

We want to show that $\int |D^i u|^p < \infty$

if $\delta < \frac{n-p}{p}$

It suffices to prove that

$$\int_\Omega |Du|^p < \infty$$

Consider $\displaystyle\int_{\Omega \setminus B(0,\varepsilon)} |Du|^p \, dx$

$$= \int_{\Omega \setminus B(0,\varepsilon)} \left(\frac{\delta}{|x|^{\delta+1}} \right)^p dx$$

$$= n \alpha_n \int_\varepsilon^1 \delta^p \, r^{-p(\delta+1)} \cdot r^{n-1} \, dr$$

$$= n \alpha_n \delta^p \int_\varepsilon^1 r^{n-p(\delta+1)-1} \, dx$$

$$= n \alpha_n \delta^p \frac{r^{n-p(\delta+1)}}{n-p(\delta+1)} \Bigg]_\varepsilon^1$$

$\Rightarrow \displaystyle\lim_{\varepsilon \to 0} \int_{\Omega \setminus B(0,\varepsilon)} |Du|^p dx < \infty$ if $n - p(\delta+1) > 0$

$\Rightarrow u \in W^{1,p}$ if $n - p(\delta+1) > 0$

Ex. Let $\{r_k\}_{k=1}^{\infty}$ be a countable dense subset in $B(0,1)$.

Let $$w(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\delta}$$

where $\delta < \frac{n-p}{p}$

Then $w \in W^{1,p}$ and

w is unbounded on

any open set in $B(0,1)$

pf: $v_k = |x - r_k|^{-\delta} \in W^{1,p}$

from previous example

b/c $W^{1,p}$ is a Banach space

$$\Rightarrow u_k = \sum_{l=1}^{k} \frac{1}{2^l} \cdot |x - r_l|^{-\delta} \in W^{1,p}$$

$k > m$

$$\Rightarrow \left\| u_k - u_m \right\|_{W^{1,p}} \leq \sum_{l=m+1}^{k} \frac{1}{2^l} \left\| |x - r_l|^{-\delta} \right\|_{W^{1,p}}$$

$$\leq C \cdot \sum_{l=m+1}^{k} \frac{1}{2^l}$$

$\Rightarrow$ u_k is a cauchy seq.

$\Rightarrow$ $\lim_{k\to\infty} u_k$ converge in $W^{1,p}$

$\| |x - r_l|^{-\delta} \|_{W^{1,p}} \leq C$

indep of l

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\delta} \in W^{1,p}$$

if $\frac{n-p}{p} > \delta$

PDE NOTES
on Oct.27
6 pages

Problem 2 from HW2.

A $C^0(\Omega)$ ftn u is subharmonic in Ω if for every ball $B \subset\subset \Omega$ and every function h harmonic in B satisfying $u \le h$ on ∂B

$\Longrightarrow$ $u \le h$ in B

Prove that a $C^0(\Omega)$ subharmonic ftn satisfying the strong maximum principle.

pf: We want to show that if u achieves an interior maximum $\Rightarrow$ $u \equiv$ constant in Ω

Suppose $\underline{x_0}$ achieves its interior maximum

$\Rightarrow$ We want to show that $u \equiv$ constant on $B(x_0, r) \subset\subset \Omega$

$$\Rightarrow \left(\begin{array}{l} u \equiv \text{const in } \Omega \\ u(x_0) = \underset{\sup_\Omega u}{\overset{\shortparallel}{M}} \Rightarrow u^{-1}(M) \text{ is open} \\ \qquad\qquad\qquad\quad \text{and closed} \end{array} \right)$$

1

If $u \not\equiv$ const in $B(x_0, r)$

then we can find a r_0 s.t.

$u\big|_{\partial B(x_0, r_0)} \not\equiv$ constant.

Consider h to be the harmonic fcn

when $\begin{cases} \Delta h = 0 & \text{in } B(x_0, r_0) \\ h\big|_{\partial B(x_0, r_0)} = u & (\text{cts}) \end{cases}$

In particular, $u \le h$ in $B(x_0, r_0)$

(b/c u is $C^0(\Omega)$ subharmonic fcn)

$\Rightarrow$ $u(x_0) \le h(x_0)$ *

Since h is harmonic

$\Rightarrow$ (maximum principle) $\max\limits_{\partial B(x_0, r_0)} h(x) = \max\limits_{B(x_0, r_0)} h$

$\Rightarrow$ ** $h(x_0) \le \max\limits_{B(x_0, r_0)} h = \max\limits_{\partial B(x_0, r_0)} h(x)$

$= \max\limits_{\partial B(x_0, r_0)} u(x) \le u(x_0)$

b/c $u(x_0) = \sup\limits_{x \in \Omega} u$

From * and **

$\Rightarrow$ $u(x_0) = h(x_0) = \max\limits_{B(x_0, r_0)} h$

b/c h is harmonic

By strong maximum principle $\Rightarrow$ $h \equiv$ constant in $B(x_0, r_0)$

$\Rightarrow$ $h\big|_{\partial B(x_0, r_0)} = u\big|_{\partial B(x_0, r_0)} \equiv$ constant

$\rightarrow\leftarrow$ we assume $\not\equiv$ constant.

§5.3 Approximation

§5.3.1 Interior approximation by smooth ftns.

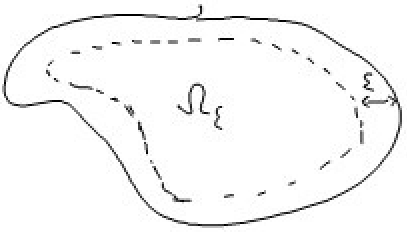
Goal: Given $u \in W^{k,p}(\Omega)$.

Then $\exists \{u_m\}_{m=1}^{\infty} \in C^{\infty}(\Omega) \cap W^{k,p}(\Omega)$

s.t $\|u_m - u\|_{W^{k,p}(\Omega)} \longrightarrow 0$

Recall $\Omega \subset \mathbb{R}^n$ open (bounded). $\varepsilon > 0$

① $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$



$\Omega_\varepsilon \subset\subset \Omega$

② $\eta(x)$ is a radial ftn with

(a) $\int_{\mathbb{R}^n} \eta(x) = 1$

(b) $\eta(x) \geq 0$

(c) $\eta(x) > 0$ when $|x| < 1$

(d) $\eta(x) = 0$ when $|x| \geq 1$

Let $\eta_\epsilon(x) = \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right)$.

Then $\int_{\mathbb{R}^n} \eta_\epsilon(x)\,dx = 1$,

$\eta_\epsilon(x) = 0$ when $|x| \geq \epsilon$

$\eta_\epsilon(x) > 0$ when $|x| < \epsilon$

Def: If $f : \Omega \to \mathbb{R}$ and $f \in L^1_{loc}(\Omega)$

define $f^\epsilon(x) = (\eta_\epsilon * f)(x)$ in Ω_ϵ

$$\left(= \int_\Omega \eta_\epsilon(x-y) f(y)\,dy = \int_{B(0,\epsilon)} \eta_\epsilon(y) f(x-y)\,dy \right.$$

$(x \in \Omega_\epsilon,\ y \in B(0,\epsilon) \Rightarrow x-y \in \Omega)$

Properties of mollification.

(i) $f^\epsilon \in C^\infty(\Omega_\epsilon)$

(ii) $f^\epsilon \to f$ a.e. as $\epsilon \to 0$

(iii) If $f \in C^0(\Omega)$ then

$f^\epsilon \to f$ uniformly on cpt subsets of Ω.

(iv) If $1 \leq p < \infty$ and $f \in L^p_{loc}(\Omega)$

$\Longrightarrow f^\epsilon \to f$ in $L^p_{loc}(\Omega)$

Th Assume $u \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$ and set $u^\epsilon = \eta_\epsilon * u$ in Ω_ϵ

Then (i) $u^\epsilon \in C^\infty(\Omega_\epsilon)$

(ii) $u^\epsilon \longrightarrow u$ in $W^{k,p}_{loc}(\Omega)$

as $\epsilon \to 0$

pf: 1° (i) follows from the property of mollification.

2°

$$u_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y)\, u(y)\, dy$$

$$\left(= \int_{B(0,\epsilon)} \eta_\epsilon(y)\, u(x-y)\, dy \right)$$

$$D^\alpha u^\epsilon(x) = \int D_x^\alpha\big(\eta_\epsilon(x-y)\big)\, u(y)\, dy$$

Since η_ϵ is radial symmetry

$$\Rightarrow \quad D_y^\alpha \eta_\epsilon(x-y) = (-1)^{|\alpha|} D_x^\alpha \eta_\epsilon(x-y)$$

$$\Rightarrow D^\alpha u^\epsilon(x) = (-1)^{|\alpha|} \underline{\int D_y^\alpha (\eta_\epsilon(x-y))\, u(y)\, dy}$$

$C_0^\infty(\Omega)$

$u \in W^{k,p}$

integration by parts

$$= (-1)^{|\alpha|} \cdot (-1)^{|\alpha|} \underline{\int_\Omega D^\alpha u(y)\, \eta_\epsilon(x-y)\, dy}$$

$$= \int_\Omega \eta_\epsilon(x-y)\, D^\alpha u(y)\, dy$$

$$= (\underline{D^\alpha u})^\epsilon$$

$L^p(\Omega)$

$$\underset{\text{Smooth}}{D^\alpha(u^\epsilon)} = \underset{L_p(\Omega)}{(D^\alpha u)^\epsilon}$$

Let $V \subset\subset \Omega$.

Then $\underline{D^\alpha(u^\epsilon)} = (D^\alpha u)^\epsilon \longrightarrow D^\alpha u$ in $L_p(V)$

$$\|u^\epsilon - u\|_{W^{k,p}(V)} = \sum_{|\alpha| \le k} \|D^\alpha(u_\epsilon) - D^\alpha(u)\|_{L^p(V)}$$

$\downarrow \epsilon \to 0$

0

$u^\epsilon \longrightarrow u$ in $W^{k,p}(V)$ as $\epsilon \longrightarrow 0$

$\Rightarrow u^\epsilon \longrightarrow u$ in $W^{k,p}_{loc}(\Omega)$

In problem 4 (HW3).

PDE NOTES
on Oct.29
6 pages

$f'(0) = 0$

$$\Longleftrightarrow \int_\Omega \underbrace{\frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla u|^2}}}_{= \vec{X}\cdot\nabla\phi} = 0 \quad \text{where } \phi \in C_0^\infty(\Omega)$$

Recall $\vec{X} \cdot \nabla \phi$

$$= \operatorname{div}(\vec{X}\phi) - \operatorname{div}(\vec{X})\phi$$

$$\left(\text{b/c} \quad \operatorname{div}(\vec{X}\phi) = \operatorname{div}(\vec{X})\phi + \vec{X}\cdot\nabla\phi\right.$$

$$0 = \int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\phi\right) - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\phi$$

$$= \underbrace{\int_{\partial\Omega}\left(\frac{\nabla u\, \phi}{\sqrt{1+|\nabla u|^2}}\right)\cdot \nu}_{=0 \text{ b/c } \phi \in C_0^\infty(\Omega)} - \int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\phi$$

$$\Rightarrow \int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\cdot \phi = 0$$

$$\Rightarrow \underbrace{\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0}_{\text{Eq for minimal graph.}}$$

problem 2 (HW3)

Show that the Green's ftn
$G(x,y) > 0$ in Ω.

pf: Recall that if u is harmonic in Ω (bounded) and $u \in C^0(\bar{\Omega})$
then either
$$\inf_{\partial\Omega} u < u < \sup_{\partial\Omega} u.$$
or $u \equiv$ constant.

Recall that
$$G(x,y) = \Phi(x,y) - \underline{\underline{\phi^x(y)}}$$
and
$$\begin{cases} \Delta_y \phi^x(y) = 0 \\ \phi^x(y)\big|_{\partial\Omega} = \Phi(x,y) \end{cases}$$

fix x

Let $u(z) = \Phi(x,z) - \phi^x(z)$

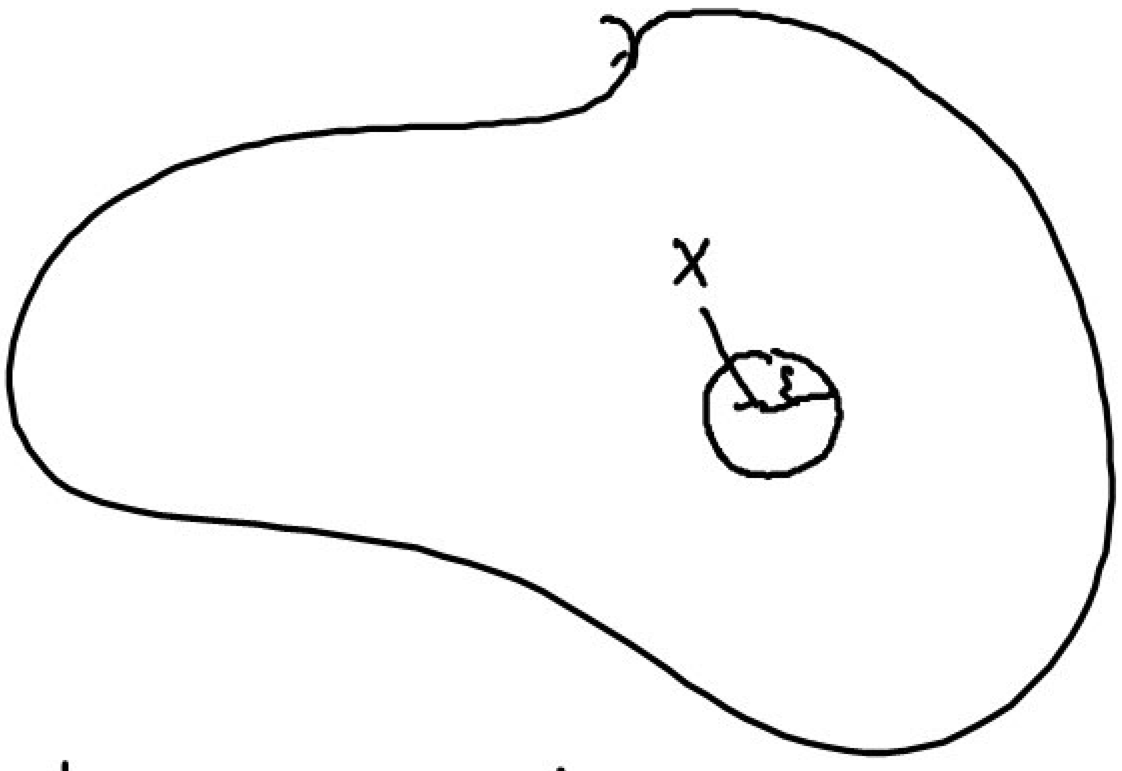
Then $\Delta u = 0$ if $z \neq x$

$\Delta_z \phi^x(z) = 0$ for all z.

$\phi^x(z)$ is a bounded ftn in Ω

$\lim_{z \to x} \Phi(x,z) = \infty$ (depends only on $\|x-z\|$)

$\Rightarrow \lim_{z \to x} u(z) = \lim_{z \to x} \Phi(x,z) - \underbrace{\phi^x(z)}_{\text{bounded}} = \infty$



So we can find ε small enough s.t

$u(z) > 0$ on $\partial B(x,\varepsilon)$

Recall that $\underset{\displaystyle \Phi(x,z)-\phi^x(z)}{\underset{\shortparallel}{u(z)}} = 0$ on $\partial\Omega$.

So $\begin{cases} \Delta_z u = 0 & \text{in } \underline{\Omega \setminus B(x,\varepsilon)} \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{on } \partial B(x,\varepsilon) \end{cases}$

$\Rightarrow$ $0 \le u$ on $\partial\left(\Omega \setminus B(x,\varepsilon)\right)$

and $u \not\equiv$ constant

$\Rightarrow$ By strong maximum principle

$u > 0$ in $\Omega \setminus B(x,\varepsilon)$

$\Rightarrow$ $G(x,z) > 0$ for $z \in \Omega \setminus B(x,\varepsilon)$

This is true for all ε small enough

$G(x,z) > 0$ for all $z \in \Omega \setminus \{x\}$.

#.

Last time, we proved that

if $u \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$

Let $V \subset\subset \Omega$.

Then $\lim\limits_{\varepsilon \to 0} \|u^\varepsilon - u\|_{W^{k,p}(V)} = 0$

$u^\varepsilon \in C^\infty(\Omega_\varepsilon)$.

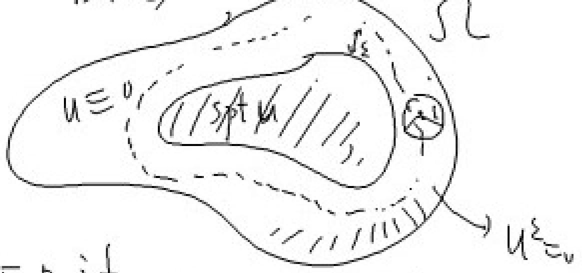
Remark: If $u \in W^{k,p}(\Omega)$ and

u has compact support in Ω

then $\lim\limits_{\varepsilon \to 0} \|u^\varepsilon - u\|_{W^{k,p}(\Omega)} = 0$

$u^\varepsilon \in C_0^\infty(\Omega)$ for ε small enough.

pf: $u^\varepsilon(x) = \displaystyle\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\, u(x-y)\, dy$



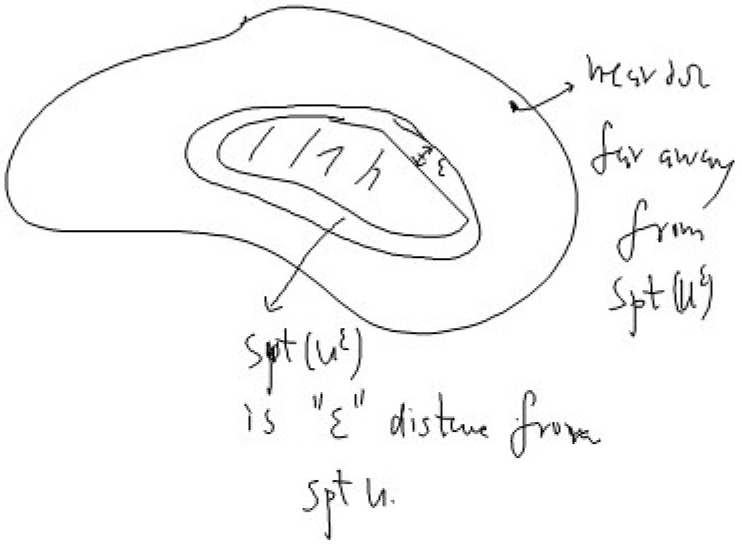
$u(x-y) = 0$ if $x-y \in \Omega \setminus (\operatorname{spt} u)$

So $u^\varepsilon(x) = 0$ if ε is small enough

and x is close to the boundary of Ω.

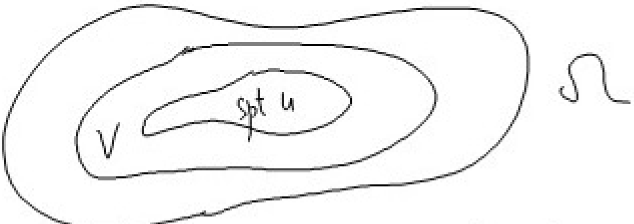
$\Rightarrow$ u^ε has compact support in Ω

for ε small enough



Choose $V \subset\subset \Omega$

and $\operatorname{spt} u \subset\subset V$



By previous Th $\|u^\varepsilon - u\|_{W^{1,p}(V)} \overset{\text{as } \varepsilon \to 0}{=} 0$

Also u^ε has compact support in V if ε is small enough.

Since u^ε and u has compact support in V

$$\Rightarrow \quad \|u^\varepsilon - u\|_{W^{k,p}(V)} = \|u^\varepsilon - u\|_{W^{k,p}(\Omega)}$$

($u^\varepsilon = u = 0$ on $\Omega \setminus V$)

So $u^\varepsilon \in C_0^\infty(\Omega)$ and

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - u\|_{W^{k,p}(\Omega)} = 0$$

Th (Partition of Unity)

Let Ω be an open set and

$\Omega \subset \bigcup_{i=1}^{\infty} V_i$ when V_i is open

and $\overline{V_i}$ is compact in Ω.

$(V_i \subset\subset \Omega)$

Then there exists a seq of ftn $\{\phi_i\}_{i=1}^{\infty}$ s.t

(i) $0 \le \phi_i \le 1$ and $\phi_i \in C_0^{\infty}(V_i)$

(ii) For each $x \in \Omega$,

there are only finitely many i

such that $\phi_i(x) \neq 0$

(iii) $\sum_{i=1}^{\infty} \phi_i(x) = 1$ for all $x \in \Omega$

(from (ii) this is only a finite sum)

Let $u \in W^{k,p}(\Omega)$

Th: $\exists$ $u_m \in C^{\infty}(\Omega) \cap W^{k,p}(\Omega)$ s.t

$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$$

Th 2 (on p 251)
(Global approximation by smooth ftns)

Assume Ω is bounded.
Let $u \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$.
Then there exist ftns $\{u_m\}_{m=1}^{\infty}$
$u_m \in C^{\infty}(\Omega) \cap W^{k,p}(\Omega)$ such that

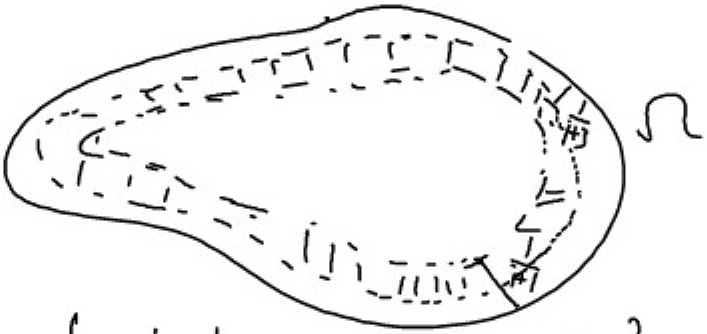
$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$$

Remark: $C^{\infty} \cap W^{k,p}(\Omega) \underset{\text{dense}}{\hookrightarrow} W^{k,p}(\Omega)$

pf: Let $U_i = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{1}{i}\}$.

$\Rightarrow \Omega = \bigcup_{i=1}^{\infty} U_i$.

Let $V_i = U_{i+3} - \overline{U}_{i+1} = \{x \mid \frac{1}{i+3} < \operatorname{dist}(x, \partial\Omega) < \frac{1}{i+1}\}$

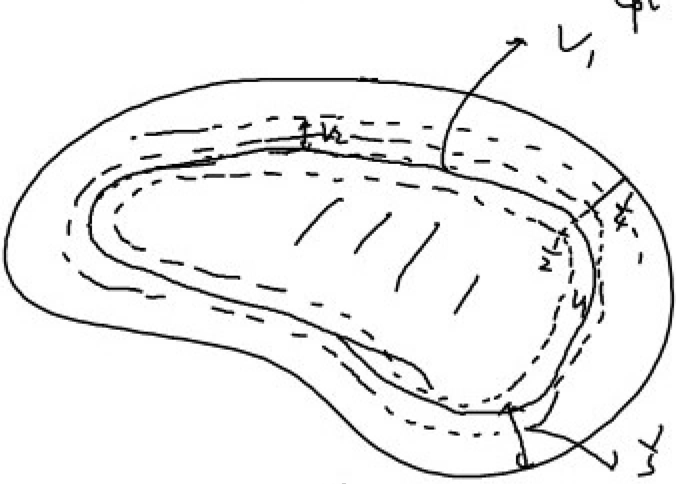


$V_1 = \left\{ x \mid \frac{1}{4} < dist(x, \partial\Omega) < \frac{1}{2} \right\}$.

$V_0 = \left\{ x \mid dist(x, \partial\Omega) > \frac{1}{3} \right\}$ $(\Rightarrow \overline{V_0} \subset\subset \Omega)$

Then $\Omega \subset \bigcup_{i=0}^{\infty} V_i$ and $\overline{V_i} \subset\subset \Omega$

cpt



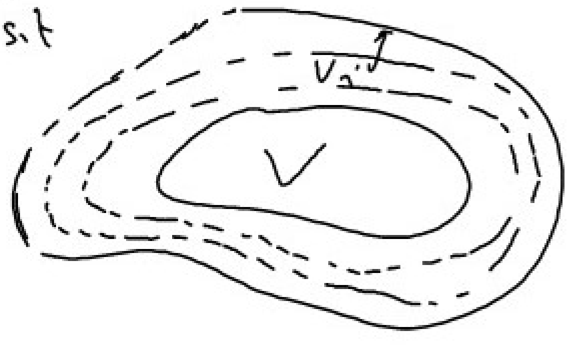
$V_1, V_2, \cdots$ are "annular" region

and getting close to the $\partial\Omega$.
as $i \to \infty$

In particular, given any open set

$V \subset\subset \Omega$

Then $\exists N$ s.t. $V \cap V_i = \phi$ for $i \geq N$



Now let $\{\phi_i\}_{i=0}^{\infty}$ be a smooth partition of unity subordinate to the open covering $\{V_i\}_{i=0}^{\infty}$, i.e.

$$\begin{cases} 0 \le \phi_i \le 1, \quad \phi_i \in C_0^\infty(V_i) \\ \underbrace{\sum_{i=0}^{\infty} \phi_i(x) = 1}_{\text{only a finite sum}} \quad \text{for } x \in \Omega \end{cases}$$

Now given $u \in W^{k,p}(\Omega)$

By Th 1 (iv) in §5.2,

$\Rightarrow \phi_i u \in W^{k,p}(\Omega)$

and $\operatorname{spt}(\phi_i u) \subset V_i \subset\subset \Omega$

By Th (we proved last time), given any $\delta > 0$, $\exists$ ε_i small enough s.t. $u^i = \eta_{\varepsilon_i} * (\phi_i u)$ and

$$\| u^i - \phi_i u \|_{W^{k,p}(\Omega)} \le \frac{\delta}{2^{i+1}}$$

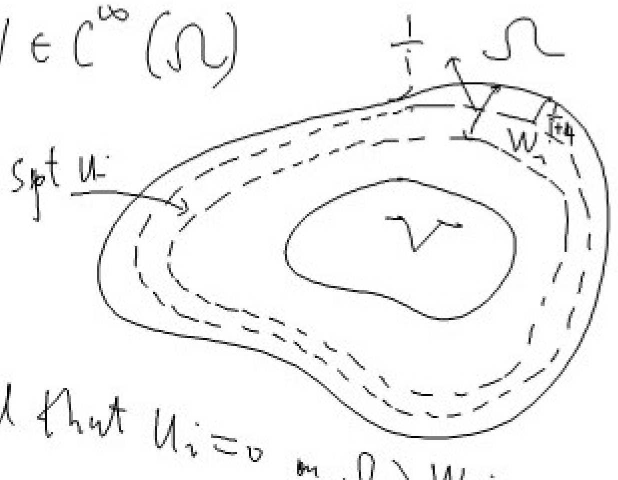
$$\operatorname{spt} u_i \subset W_i = U_{i+4} - \overline{U_i}$$

$$\left(\sup(\phi_i) \subset V_i = U_{i+3} - \overline{U_{i+1}} \right)$$

$U_{i+3} - U_{i+1})$

Let $v = \sum_{i=0}^{\infty} u^i = \sum_{i=0}^{\infty} J_{\varepsilon_i} * (\phi_i u)$

claim $v \in C^\infty(\Omega)$



Recall that $u_i = 0$ on $\Omega \setminus W_i$

Since $V \subset\subset \Omega$, $V \cap W_i \neq \emptyset$ for only a finite number of index i,

$\Rightarrow \quad x \in V \Rightarrow u_i(x) \neq 0$ for only a finite # of index

$x \in V$, $v(x) = \sum_i u_i(x)$ is only a finite sum.

$\Rightarrow$ v is smooth in $V \subset\subset \Omega$

$\Rightarrow$ v is smooth in Ω

Since $u = \sum_{i=0}^{\infty} \phi_i u \quad \left(\sum_{i=0}^{\infty} \phi_i = 1\right)$

$$\Rightarrow \|v - u\|_{W^{k,p}(V)} = \left\|\sum_{i=0}^{\infty} (u_i - \phi_i u)\right\|_{W^{k,p}(V)}$$

$$\le \sum_{i=0}^{\infty} \|u^i - \phi_i u\|_{W^{k,p}(\Omega)}$$

$$\le \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} < \delta$$

$$\Longrightarrow \quad \|v-u\|_{W^{1,p}(\Omega)} \leq \delta$$

$$v \in C^{\infty}(\Omega) \cap W^{1,p}$$

#.

§ 5.6 Sobolev Inequalities

$$W^{1,p}(\Omega) = \{ u \mid u, |Du| \in L^p(\Omega) \}$$

PDE NOTES
on Nov.3
6 pages

$$W^{1,p}(\Omega) \overset{?}{\hookrightarrow} L^q(\Omega)$$

We want to see if

$$\|u\|_{L^q(\Omega)} \leq C \|Du\|_{L^p(\Omega)}$$

where C depends only on p, n, Ω.

In the following, we'll prove that

$$\|u\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Motivation:

Find out the possible exponent

where

$$\circledast \quad \|u\|_{L^q(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

for $u \in C_0^\infty(\mathbb{R}^n)$

Suppose ~~*~~ holds for u.

Let $u_\lambda(x) = u(\lambda x)$ where $\lambda > 0$

Since $u \in C_0^\infty(\mathbb{R}^n)$, $u_\lambda \in C_0^\infty(\mathbb{R}^n)$

So $\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C \|Du_\lambda\|_{L^p(\mathbb{R}^n)}$

Compute $\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \left(\int |u_\lambda(x)|^q dx\right)^{\frac{1}{q}}$

$= \left(\int_{\mathbb{R}^n} |u(\lambda x)|^q dx\right)^{\frac{1}{q}}$: $y = \lambda x$, $\frac{dy}{\lambda^n} = dx$

$= \left(\int_{\mathbb{R}^n} \frac{|u(y)|^q dy}{\lambda^n}\right)^{\frac{1}{q}}$

$= \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}$

$Du_\lambda(x) = D(u(\lambda x)) = (Du)(\lambda x) \cdot \lambda$

$\Rightarrow \|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |Du_\lambda(x)|^p dx\right)^{\frac{1}{p}}$

$= \left(\int_{\mathbb{R}^n} |(Du)(\lambda x)|^p \cdot \lambda^p dx\right)^{\frac{1}{p}}$ $\boxed{\begin{array}{l} y = \lambda x \\ dx = \frac{1}{\lambda^n} dy \end{array}}$

$= \left(\int |Du(y)|^p \frac{\lambda^p}{\lambda^n} dy\right)^{\frac{1}{p}}$

$= \lambda^{\frac{p-n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$

We have

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}$$

$$\|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{\frac{p-n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C \|Du_\lambda\|_{L^p(\mathbb{R}^n)}$$

$$\Longleftrightarrow \quad \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \quad \|u\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{1-\frac{n}{p}+\frac{n}{q}} \|Du\|_{L^p(\mathbb{R}^n)}$$

If $1-\frac{n}{p}+\frac{n}{q} > 0$

then $\lim_{\lambda\to\infty} \lambda^{1-\frac{n}{p}+\frac{n}{q}} = \infty$

If $1-\frac{n}{p}+\frac{n}{q} < 0$

then $\lim_{\lambda\to 0^+} \lambda^{1-\frac{n}{p}+\frac{n}{q}} = \infty$

So $\|u\|_{L^q(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$

If $1 - \frac{n}{p} + \frac{n}{q} = 0$

$\iff \frac{n}{q} = \frac{n}{p} - 1$

$\iff \frac{1}{q} = \frac{1}{p} - \frac{1}{n} \left(= \frac{n-p}{np} \right)$

$\iff q = \frac{np}{n-p}$

Def: If $1 \le p < \infty$

the Sobolev conjugate of p

is $p^* = \frac{np}{n-p}$. Note that

$\frac{1}{p^*} + \frac{1}{n} = \frac{1}{p}$.

Th2 (Gagliardo-Nirenberg Sobolev Inequality)

Assume $1 \le p < n$, $\exists$ a constant C

depending only on p, n s.t

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_0^1(\mathbb{R}^n)$.

pf: First, prove the case when $p=1$

$\Rightarrow p^* = \frac{n}{n-1} \quad \left(p^* = \frac{np}{n-p} \right)$

We want to show that

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \le C \left(\int_{\mathbb{R}^n} |Du| dx \right)$$

$$\Leftrightarrow \left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \right) \le C \left(\int_{\mathbb{R}^n} |Du| \right)^{\frac{n}{n-1}}$$

Since u has compact support,

$$u(x_1, \dots x_{i-1}, z_i, x_{i+1}, \dots x_n) = 0$$

when $\|z_i\|$ is large

$$\Rightarrow u(x) = \int_{-\infty}^{x_i} \frac{\partial u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots x_n)}{\partial x_i} dy_i$$

$$\Rightarrow |u(x)| \le \int_{-\infty}^{\infty} \underbrace{\left| \frac{\partial u(x_1, \dots x_{i-1}, y_i, x_{i+1}, \dots, x_n)}{\partial x_i} \right|}_{\text{indep of } x_i} dy_i$$

$$\Rightarrow |u(x)|^{\frac{1}{n-1}} \le \left(\int_{-\infty}^{\infty} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)| dy_i \right)^{\frac{1}{n-1}}$$

$$\Rightarrow |u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(\dots, y_i, \dots)| \, dy_i \right)^{\frac{1}{n-1}}$$

$$\Rightarrow \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1$$

$$\leq \int_{-\infty}^{\infty} \left(\underbrace{\int_{-\infty}^{\infty} |Du(y_1, x_2, \cdots x_n)| dy_1}_{\text{indep of } x_1} \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du(\cdots y_i \cdots)| dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{-\infty}^{\infty} |Du(y_1, x_2, \cdots x_n)| dy_1 \right)^{\frac{1}{n-1}} \cdot \int_{-\infty}^{\infty} \prod_{i=2}^{n} \boxed{\left(\int_{-\infty}^{\infty} |Du(\cdots y_i \cdots)| dy_i \right)^{\frac{1}{n-1}}} dx_1$$

$$\left(\int_{\mathbb{R}} \prod_{i=2}^{n} |f_i(z)| dz \leq \prod_{i=2}^{n} \|f_i\|_{L^{n-1}(\mathbb{R})} \right.$$

$f_i \;\|\; \left(\int |f_i|^{n-1} \right)^{\frac{1}{n-1}}$

$$\underbrace{\frac{1}{n-1} + \cdots + \frac{1}{n-1}}_{n-1 \text{ terms}} = 1$$

Holder Inequality.

$f_i \in L^{n-1}(\mathbb{R})$

$$\leq \left(\int_{-\infty}^{\infty} |Du(y_1, x_2, \cdots x_n)| dy_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |Du(\cdots, y_i, \cdots)| dy_i dx_1 \right)^{\frac{1}{n-1}}$$

Holder

PDE NOTES
on Nov.5
6 pages

Next, we want to establish Sobolev inequality for $W^{k,p}(\Omega)$.

We study the case for $W_0^{k,p}(\Omega)$.

Recall that

① $W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$ in $W^{k,p}(\Omega)$

So given any $u \in W_0^{k,p}(\Omega)$

$\exists \{u_m\}_{m=1}^\infty$ with $u_m \in C_0^\infty(\Omega)$

s.t $\lim_{k \to \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$

② Suppose Ω is bounded

$1 \le q < p$, $u \in L^p(\Omega)$

$$\Longrightarrow \|u\|_{L^q(\Omega)} \le |Vol(\Omega)|^{\frac{1}{q}-\frac{1}{p}} \|u\|_{L^p(\Omega)}$$

$$L^q(\Omega) \qquad\qquad L^p(\Omega)$$

pf: $\left(\int u^q \cdot 1\right) \le \left(\int (u^q)^{\frac{p}{q}}\right)^{\frac{q}{p}} \left(\int_\Omega 1^{\cdots}\right)^{\frac{q}{p}-1}$

$$\left(\int u^q\right) \le \left(\int u^p\right)^{\frac{q}{p}} (\mathrm{Vol}(\Omega))^{\frac{q}{p}-1}$$

$$\Rightarrow \quad \|u\|_{L^q(\Omega)} \le (\mathrm{Vol}(\Omega))^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^p(\Omega)}$$

Th3 on p265

Assume Ω is a bounded, open subset of $\mathbb{R}^n$.

Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \le p < n$.

Then $\|u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{p^*}-\frac{1}{q}} \dfrac{p(n-1)}{n-p} \|Du\|_{L^p(\Omega)}$

for each $1 \le q \le p^*$

$|\Omega| = \mathrm{Vol}(\Omega)$.

Theorem 1 p263 (Gagliardo-Nirenberg-Sobolev inequality)
Assume $1 \leq p < n$. There exists a constant C, depending only on p and n, such that

$$||u||_{L^{\frac{np}{n-p}}(R^n)} \leq C||Du||_{L^p(R^n)}$$

for all $u \in C_0^1(R^n)$.
Remark: From the proof, we my choose $C(n,p) = \frac{p(n-1)}{n-p}$. But this may not be the best constant.

Proof:

1. First, we prove the case $p = 1$. We want to show that

$$||u||_{L^{\frac{n}{n-1}}(R^n)} \leq C||Du||_{L^1(R^n)}$$

for all $u \in C_0^1(R^n)$. Since u has compact support, we have for each $1 \leq i \leq n$ and $x \in R^n$

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)dy_i$$

and

$$\begin{aligned} |u(x)| &\leq \int_{-\infty}^{x_i} |u_{x_i}(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|dy_i \\ &\leq \int_{-\infty}^{x_i} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|dy_i \\ &\leq \underbrace{\int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|dy_i}_{independent\ of\ x_i} . \end{aligned} \tag{1}$$

Consequently,

$$|u(x)|^{\frac{1}{n-1}} \leq (\int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|dy_i)^{\frac{1}{n-1}}$$

and

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \underbrace{(\int_{-\infty}^{\infty} |Du(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|dy_i)^{\frac{1}{n-1}}}_{independent\ of\ x_i}$$

2. To illustrate the main ideas, we discuss the case when $n = 3$. So we have

$$|u(x)|^{\frac{3}{2}} \le (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3)^{\frac{1}{2}}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty} |u(x)|^{\frac{3}{2}} dx_1 \\
\le\ & \int_{-\infty}^{\infty} \underbrace{(\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}}}_{independent\ \ of\ \ x_1} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3)^{\frac{1}{2}} dx_1 \\
=\ & (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}} \int_{-\infty}^{\infty} \underbrace{(\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2)^{\frac{1}{2}} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3)^{\frac{1}{2}} dx_1}_{apply\ \ holder\ \ inequality\ \ w.r.t.\ \ x_1\ \ \int |fg| dx_1 \le (\int f^2 dx_1)^{\frac{1}{2}} (\int g^2 dx_1)^{\frac{1}{2}}} \\
\le\ & (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}} \\
& \underbrace{(\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2) dx_1)^{\frac{1}{2}}}_{independent\ \ of\ \ x_2} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3) dx_1)^{\frac{1}{2}}
\end{aligned} \tag{2}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)|^{\frac{3}{2}} dx_1 dx_2 \\
\le\ & (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2) dx_1)^{\frac{1}{2}} \\
& \int_{-\infty}^{\infty} \underbrace{(\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1)^{\frac{1}{2}} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3) dx_1)^{\frac{1}{2}} dx_2}_{apply\ \ holder\ \ inequality\ \ w.r.t.\ \ x_2} \\
\le\ & (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2) dx_1)^{\frac{1}{2}} \\
& (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1) dx_2)^{\frac{1}{2}} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, x_2, y_3)| dy_3) dx_1) dx_2)^{\frac{1}{2}} \\
=\ & ||Du||_{L^1(R^3)}^{\frac{1}{2}} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(x_1, y_2, x_3)| dy_2) dx_1)^{\frac{1}{2}} (\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} |Du(y_1, x_2, x_3)| dy_1) dx_2)^{\frac{1}{2}}
\end{aligned} \tag{3}$$

Integrating w.r.t x_3, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|u(x)|^{\frac{3}{2}}dx_1dx_2dx_3\\
&\leq ||Du||^{\frac{1}{2}}_{L^1(R^3)}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|Du(x_1,y_2,x_3)|dy_2)dx_1)^{\frac{1}{2}}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|Du(y_1,x_2,x_3)|dy_1)dx_2)^{\frac{1}{2}}dx_3\\
&\leq ||Du||^{\frac{1}{2}}_{L^1(R^3)}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|Du(x_1,y_2,x_3)|dy_2)dx_1)dx_3)^{\frac{1}{2}}\\
&\quad(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}|Du(y_1,x_2,x_3)|dy_1)dx_2)dx_3)^{\frac{1}{2}}\\
&= ||Du||^{\frac{3}{2}}_{L^1(R^3)}.
\end{aligned}
\tag{4}
$$

Thus we have

$$||u||^{\frac{3}{2}}_{L^{\frac{3}{2}}(R^3)} \leq ||Du||^{\frac{3}{2}}_{L^1(R^3)}.$$

This implies

$$||u||_{L^{\frac{3}{2}}(R^3)} \leq ||Du||_{L^1(R^3)}.$$

3. For the general case, we start with

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n}(\underbrace{\int_{-\infty}^{\infty}|Du(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_i}_{independent\ of\ x_i})^{\frac{1}{n-1}}.$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}|u(x)|^{\frac{n}{n-1}}dx_1\\
&\leq \int_{-\infty}^{\infty}\underbrace{(\int_{-\infty}^{\infty}|Du(y_1,x_2,\cdots x_n)|dy_1)^{\frac{1}{n-1}}}_{independent\ of\ x_1}\prod_{i=2}^{n}(\underbrace{\int_{-\infty}^{\infty}|Du(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_i}_{independent\ of\ x_i})^{\frac{1}{n-1}}dx_1\\
&= (\int_{-\infty}^{\infty}|Du(y_1,x_2,\cdots x_n)|dy_1)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\underbrace{\prod_{i=2}^{n}(\int_{-\infty}^{\infty}|Du(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_i)^{\frac{1}{n-1}}dx_1}_{apply\ holder\ inequality\ w.r.t.\ x_1\ \ \int|f_2\cdots f_n|dx_1\leq\prod_{i=2}^{n}(\int f_i^{n-1}dx_1)^{\frac{1}{n-1}}}\\
&\leq (\int_{-\infty}^{\infty}|Du(y_1,x_2,\cdots x_n)|dy_1)^{\frac{1}{n-1}}\prod_{i=2}^{n}(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)|dy_idx_1)^{\frac{1}{n-1}}
\end{aligned}
\tag{5}
$$

Repeating this process, we get

$$\int_{R^n} |u(x)|^{\frac{n}{n-1}} dx \le (\int_{R^n} |Du| dx)^{\frac{n}{n-1}}.$$

Thus

$$||u||^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(R^n)} \le ||Du||^{\frac{n}{n-1}}_{L^1(R^n)}.$$

This implies

$$|||u||_{L^{\frac{n}{n-1}}(R^n)} \le ||Du||_{L^1(R^n)}. \tag{6}$$

4. Now we consider the case $1 < p < n$. Let $v := |u|^\gamma = (u^2)^{\frac{\gamma}{2}}$ where $\gamma > 1$ is to be determined. Then $v_{x_i} = \frac{\gamma}{2}(u^2)^{\frac{\gamma}{2}-1}(2uu_{x_i}) = \gamma(u^2)^{\frac{\gamma}{2}-1}(uu_{x_i})$ and $|Dv| = \gamma|u|^{\gamma-1}|Du|$. We apply estimate 6 to $v := |u|^\gamma$ to get

$$(\int_{R^n} |v(x)|^{\frac{n}{n-1}} dx)^{\frac{n-1}{n}} \le (\int_{R^n} |Dv| dx).$$

Thus we have

$$\begin{aligned} &(\int_{R^n} |u(x)|^{\frac{\gamma n}{n-1}} dx)^{\frac{n-1}{n}} \\ &\le \gamma \int_{R^n} |u|^{\gamma-1}|Du| dx \\ &\le \gamma(\int_{R^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx)^{\frac{p-1}{p}} (\int_{R^n} |Du|^p dx)^{\frac{1}{p}} \end{aligned} \tag{7}$$

Choose γ so that $\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1}$, i.e $\frac{1}{\gamma} = 1 - \frac{n(p-1)}{p(n-1)}$ and $\gamma = \frac{p(n-1)}{n-p} > 1$ (since $1 < p < n$). Now we have $\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1} = \frac{\frac{p(n-1)}{n-p}n}{n-1} = \frac{np}{n-p}$. Hence 7 can be rewritten as $(\int_{R^n} |u(x)|^{\frac{np}{n-p}} dx)^{\frac{n-1}{n}} \le \frac{p(n-1)}{n-p}(\int_{R^n} |u|^{\frac{np}{n-p}} dx)^{\frac{p-1}{p}} (\int_{R^n} |Du|^p dx)^{\frac{1}{p}}$. Note that $\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np}$. Thus we have $(\int_{R^n} |u(x)|^{\frac{np}{n-p}} dx)^{\frac{n-p}{np}} \le \frac{p(n-1)}{n-p}(\int_{R^n} |Du|^p dx)^{\frac{1}{p}}$ which is

$$||u||_{L^{\frac{np}{n-p}}(R^n)} \le \frac{p(n-1)}{n-p}||Du||_{L^p(R^n)}.$$

Recall

PDE NOTES
on Nov.10
3 pages

① Given $u \in W_0^{k,p}(\Omega)$.

$\exists \; u_m \in C_0^\infty(\Omega)$ s.t.

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$$

② Suppose Ω is bounded

$1 \le q < p$ and $u \in L^p(\Omega)$

$$\Rightarrow \quad \|u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q}-\frac{1}{p}} \|u\|_{L^p(\Omega)}$$

③ $$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)}$$

for $u \in C_0^1(\mathbb{R}^n)$

where $\quad 1 \le p < n, \; p^* = \dfrac{np}{n-p}$

$C = C(n,p)$

Th3 (on p.269)

Assume Ω is bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \le p < n$.

Then $$\|u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{p^*}-\frac{1}{q}} \frac{p(n-1)}{n-p} \|Du\|_{L^p(\Omega)}$$

for $q \in [1, p^*]$

pf: We'll prove the case where $q = p^*$ first.

Given $u \in W_0^{1,p}(\Omega)$.

$\exists \{u_m\}_{m=1}^{\infty}$ s.t. $\lim_{m\to\infty} \|u_m - u\|_{W^{1,p}(\Omega)} = 0$

$(u_m \in C_0^{\infty}(\Omega))$

In particular $\lim_{m\to\infty} \|u_m - u\|_{L^p(\Omega)} + \sum_{|\alpha|=1} \|D^\alpha u_m - Du\|_{L^p(\Omega)} = 0$

We can extend $u_m = 0$ in $\mathbb{R}^n \setminus \Omega$

Then $u_m \in C_0^{\infty}(\mathbb{R}^n)$

$u_k - u_l \in C_0^{\infty}(\mathbb{R}^n)$

$\Rightarrow \|u_k - u_l\|_{L^{p^*}(\mathbb{R}^n)} \le \frac{p(n-1)}{n-p} \|Du_k - Du_l\|_{L^p(\mathbb{R}^n)}$

By Th 1 on p2 b3) (Gagliardo-Nirenberg-Sobolev ineq)

Recall that $u_k \in C_0^{\infty}(\Omega)$

$\Rightarrow \|u_k - u_l\|_{L^{p^*}(\Omega)} \le \frac{p(n-1)}{n-p} \|Du_k - Du_l\|_{L^p(\Omega)}$

b/c $\{Du_m\}_{m=1}^{\infty}$ is a cauchy seq in $L^p(\Omega)$

$\Rightarrow \{u_m\}_{m=1}^{\infty}$ is a cauchy seq in $L^{p^*}(\Omega)$

$\Rightarrow u_m \to u$ in $L^{p^*}(\Omega)$ (b/ $u_m \to u$ in $L^p(\Omega)$)

We have

$$\|u_m\|_{L^{p^*}(\Omega)} \leq \frac{p(n-1)}{n-p} \|Du_m\|_{L^p(\Omega)}$$

$$\left(\text{b/c} \quad u_m \in C_0^\infty(\Omega) \right)$$

Since $\lim_{m\to\infty} \|u_m - u\|_{L^{p^*}(\Omega)} = 0$

and $\lim_{m\to\infty} \|Du_m - Du\|_{L^p(\Omega)} = 0$

$$\Rightarrow \quad \lim_{m\to\infty} \|u_m\|_{L^{p^*}(\Omega)} \leq \lim_{m\to\infty} \frac{p(n-1)}{n-p} \|Du_m\|_{L^p(\Omega)}$$

$$\|u\|_{L^{p^*}(\Omega)} \leq \frac{p(n-1)}{n-p} \|Du\|_{L^p(\Omega)}$$

First, we want to state a Th about the approximation of $W^{1,p}(\Omega)$.

PDE NOTES
on Nov.12
4 pages

Th1 (p254)

Assume Ω is bounded (open) and $\partial\Omega$ is C^1.

Select a bounded open set V s.t $\Omega \subset\subset V$.

Then there exists a bounded linear operator. s.t

$$E: W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^n) \text{ s.t}$$

for each $u \in W^{1,p}(\Omega)$

(i) $Eu = u$ a.e. in Ω

(ii) Eu has compact support in V

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq \underline{C(n,p,\Omega)} \|u\|_{W^{1,p}(\Omega)}$

constant depending on p, n, Ω

Def: We call Eu an extension of u to $\mathbb{R}^n$.

We'll use this Th to prove

Th 2 (Estimate for $W^{1,p}(\Omega)$ $1\le p<n$)
↑ p265.

Let Ω be a bounded, open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1

Assume $1\le p<n$ and $u\in W^{1,p}(\Omega)$

Then $u\in L^q(\Omega)$ for $1\le q\le p^*$ with the estimate

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$$

when $C=C(n,p,\Omega)$ (constant).

pf: 1° It suffice to prove the case $q=p^*$.

For $1\le q<p^*$

$$\|u\|_{L^q} \le C'(n,p,\Omega)\|u\|_{L^{p^*}(\Omega)}$$
$$\le \underbrace{C'(n,p,\Omega)\, C(n,p,\Omega)}_{\text{ConstA}(n,p,\Omega)}\|u\|_{W^{1,p}(\Omega)}$$

2° From previous Th, choose $\Omega\subset\subset V\subset\subset\mathbb{R}^n$

We have $\bar u = Eu\in W^{1,p}(\mathbb{R}^n)$

and $\bar u$ has compact support in V

$\bar u = u$ in Ω, $\|\bar u\|_{W^{1,p}(\mathbb{R}^n)} \le C(n,p,\Omega)\|u\|_{W^{1,p}(\Omega)}$

Since $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ and $\bar{u}$ has compact support in V, we can find

$\{u_m\}_{m=1}^{\infty}$ s.t. $\lim_{m\to\infty} \|u_m - \bar{u}\|_{W^{1,p}(\mathbb{R}^n)} = 0$, i.e

$$0 = \lim_{m\to\infty} \|u_m - \bar{u}\|_{L^p(\mathbb{R}^n)} + \sum_{|\alpha|=1} \|D^\alpha u_m - D^\alpha \bar{u}\|_{L^p(\mathbb{R}^n)},$$

where $u_m \in C_0^\infty(\mathbb{R}^n)$

(from Th 1 on p250)

$\Rightarrow$ This implies that $\bar{u} \in W^{1,p}_{loc}(V)$

b/c $u_k - u_\ell \in C_0^\infty(\mathbb{R}^n)$

By Gagliardo-Nirenberg-Sobolev inequality (Th 1 on p263), we have $\|u_k - u_\ell\|_{L^{p^*}(\mathbb{R}^n)} \leq \|Du_k - Du_\ell\|_{L^p(\mathbb{R}^n)}$

Since $\{Du_m\}_{m=1}^{\infty}$ is a cauchy seq in $L^p(\mathbb{R}^n)$

$\Longrightarrow$ $\{u_m\}_{m=1}^{\infty}$ is a cauchy seq in $L^{p^*}(\mathbb{R}^n)$

$\%$ $L^{p^*}(\mathbb{R}^n)$ is complete and the limit is unique $\Rightarrow \lim_{m\to\infty}\|u_m - \bar{u}\|_{L^{p^*}(\mathbb{R}^n)} = 0$

$$\Rightarrow \quad \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le \|Du_m\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \lim_{m\to\infty} \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le \lim_{m\to\infty} \|Du_m\|_{L^p(\mathbb{R}^n)}$$

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le \|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

Now $\Omega \subset \mathbb{R}^n$ and $u \equiv \bar{u}$ in Ω.

$$\|u\|_{L^{p^*}(\Omega)} \le \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le \|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

$$\left(\int_\Omega |u|^{p^*}\right)^{\frac{1}{p^*}} \quad \left(\int_{\mathbb{R}^n} |\bar{u}|^{p^*}\right)^{\frac{1}{p^*}}$$

$$= \left(\int_\Omega |u|^{p^*} + \underbrace{\int_{\mathbb{R}^n \setminus \Omega} |\bar{u}|^{p^*}}_{\ge 0}\right)^{\frac{1}{p^*}}$$

Also $\|D\bar{u}\|_{L^p(\mathbb{R}^n)} \le \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}$

$$\|D\bar{u}\|_{L^p(\mathbb{R}^n)} = \sum_{|\alpha|=1}\left(\int_\Omega |D^\alpha \bar{u}|^p\right)^{\frac{1}{p}}$$

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} = \|\bar{u}\|_{L^p(\mathbb{R}^n)} + \sum_{|\alpha|=1} \|D^\alpha \bar{u}\|_{L^p(\mathbb{R}^n)}$$

$$= \|Eu\|_{W^{1,p}(\mathbb{R}^n)} \underset{\uparrow}{\le} C(n,p,\Omega)\|u\|_{W^{1,p}(\Omega)}$$

From our choice of $\bar{u}$

$$\Rightarrow \|u\|_{L^{p^*}(\Omega)} \le C(n,p,\Omega)\|u\|_{W^{1,p}(\Omega)}$$

#.

Remark:

Th Let $u, v \in L^1_{loc}(\Omega)$

Then $v = D^\alpha u$ (weak partial derivative)

$\Longleftrightarrow$ $\exists$ a seq of $C^\infty(\Omega)$ $\{u_m\}_{m=1}^\infty$

s.t $\begin{array}{l} u_m \longrightarrow u \\ D^\alpha u_m \longrightarrow v \end{array}$ in $L^1_{loc}(\Omega)$

$\left(\begin{array}{l} \text{ie } \Omega' \subset\subset \Omega \\ \lim\limits_{m\to\infty} \int_{\Omega'} |u_m - u| = 0 \text{ and } \lim\limits_{m\to\infty} \int_{\Omega'} |D^\alpha u_m - v| = 0 \end{array} \right)$

pf: $(\Longrightarrow)$ Modify the pf of Th 2 (on p 251) and replace $\| \ \|_{W^{k,p}(\Omega)}$ by $\| \ \|_{L^1(\Omega)}$.

$(\Leftarrow)$

We want to prove that $D^\alpha u = v$

$$\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega v \phi$$

b/c $u_m \in C^\infty(\Omega)$

$$\Rightarrow \int_\Omega \underline{u_m D^\alpha \phi} = (-1)^{|\alpha|} \int_\Omega (D^\alpha u_m) \phi, \ \phi \in C_0^\infty(\Omega)$$

b/c $\phi, D^\alpha \phi$ has compact support

$$\Rightarrow \lim_{m\to\infty} \int_\Omega u_m D^\alpha \phi = \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega (D^\alpha u_m) \phi$$

$$\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega v \phi$$

b/c $\begin{array}{l} u_m \to u \\ D^\alpha u_m \to v \end{array}$ in $L^1_{loc}(\Omega)$

$\Rightarrow \begin{array}{l} u_m D^\alpha \phi \to u D^\alpha \phi \\ (D^\alpha u_m) \phi \to v \phi \end{array}$ in $L^1_{loc}(\Omega)$

$\Rightarrow D^\alpha u = v$

Last time, we proved that

If Ω is bounded, open and with C^1 boundary,

then given $u \in W^{1,p}(\Omega)$, $1 \le p < n$

we have $u \in L^q(\Omega)$ for $1 \le q \le p^*$

and $\|u\|_{L^q(\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$

$$p^* = \underbrace{\frac{np}{n-p}}_{>1} > p$$

Remk: If $p \to n^- \Rightarrow p^* \to \infty$

One "may" expect that from $\|u\|_{L^{p^*}(\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$

$$\Downarrow p \to n^-$$

to get $\|u\|_{L^\infty(\Omega)} \le C(n,\Omega) \|u\|_{W^{1,n}(\Omega)}$

But this not true from HW4 Problem 1

$\exists$ $u \in W^{1,n}(B(0,1))$ s.t.
u is not bounded.

In the case when $p > n$, we have the following.

Th 5 on p269.

Let Ω be a bounded, open subset of $\mathbb{R}^n$ and suppose $\partial\Omega$ is C^1. Assume $n < p \leq \infty$, and $u \in W^{1,p}(\Omega)$.

Then u has a version $u^* \in C^{0,\gamma}(\bar{\Omega})$, for $\gamma = 1 - \frac{n}{p}$, with the estimate

$$\|u^*\|_{C^{0,\gamma}(\bar{\Omega})} \leq C(n,p,\Omega)\|u\|_{W^{1,p}(\Omega)}$$

Def: We say u^* is a version of a given ftn u if $u = u^*$ a.e.

Def (on p240, 5.1) Hölder Spaces

A ftn u is said to be Hölder continuous with exponent γ if $|u(x) - u(y)| \leq C|x-y|^\gamma$ for all $x, y \in \Omega$.

Remark: If $\gamma = 1$, we called this is a Lipchitz cts ftn

2° A Hölder cts ftn is uniformly cts.

$\Rightarrow$ $u \in C^0(\Omega)$ + Hölder continuity

$\Rightarrow$ $u \in C^0(\bar{\Omega})$

Def: (i) If $u: \Omega \longrightarrow \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\bar{\Omega})} = \sup_{x \in \Omega} |u(x)|$$

(ii) The γ^{th}-Hölder semi-norm of $u: \Omega \longrightarrow \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\bar{\Omega})} = \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\gamma} \right\}$$

and the γ^{th}-Hölder norm is

$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + [u]_{C^{0,\gamma}(\bar{\Omega})}$$

Def: The Hölder space $C^{k,\gamma}(\bar{\Omega})$ consists of all ftns $u \in C^k(\bar{\Omega})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar{\Omega})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{\Omega})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{\Omega})}$$

Th $C^{k,\gamma}(\bar{\Omega})$ is a Banach space.

pf: It's not difficult to show this is a normed linear space.

Given $\{u_m\}_{m=1}^{\infty}$ a cauchy seq in $C^{k,\gamma}(\bar{\Omega})$

We want to show it converges in $C^{k,\gamma}(\bar{\Omega})$

$\exists C$ s.t.

b/c $[D^\alpha u_m]_{C^{0,\gamma}(\bar{\Omega})} < C$, $|\alpha| = k$

$$\Rightarrow \frac{|D^\alpha u_m(x) - D^\alpha u_m(y)|}{|x-y|^\gamma} \le C$$

$$\Rightarrow |D^\alpha u_m(x) - D^\alpha u_m(y)| \le C|x-y|^\gamma$$

$\Rightarrow \{D^\alpha_m u\}$ is equi-ct.

$\Rightarrow$ Also $\|D^\alpha u\|_{C(\bar{\Omega})} < \infty$

$\Rightarrow \{D^\alpha u_m\}$ is uniformly and equi-ct

$\Rightarrow \{D^\alpha u_m\}$ converge to Hölder ct fn

$\Rightarrow$ Use diagonal process $u_m \to u$ in $C^{k,\gamma}(\bar{\Omega})$.

Th4 (on p266) (Morrey's inequality)

Assume $n < p \le \infty$.

Then there exists a constant $C(p,n)$, such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$

where $\gamma := 1 - \frac{n}{p} > 0$.

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |u(x)| + \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\gamma}$$

Lemma: $$⨍_{B(x,r)} |u(y) - u(x)|\, dy \le C(n) \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, d$$

for $u \in C^1(B(x,2r))$

pf: Fix $w \in \partial B(0,1)$, $0 < s < r$.

$$|u(x+sw) - u(x)| = \left| \int_0^s \underbrace{\frac{d}{dt} u(x+tw)}\, dt \right|$$

$$= \left| \int_0^s \nabla u(x+tw) \cdot w\, dt \right|, \quad |w| = 1$$

$$\le \int_0^s |Du(x+tw)|\, dt$$

$$\Rightarrow \int |u(x+sw) - u(x)|\, dS$$

$$\Rightarrow \int_{\partial B(0,1)} |u(x+sw) - u(x)| \, dS(w)$$

$$\leq \int_{\partial B(0,1)} \int_0^s |Du(x+tw)| \, dt \, dS(w)$$

$$= \int_0^s \int_{\partial B(0,1)} |Du(x+tw)| \, dS(w) \, dt$$

Let $y = x + tw$

$\Rightarrow$ $dS(y) = t^{n-1} \, dS(w)$

$tw = y - x$

$t|w| = |y-x| \quad (\because \ 0 \leq t \leq s, \ |w| = 1)$

$t = |y-x|$

$$= \int_0^s \int_{\partial B(x,t)} |Du(y)| \frac{dS(y)}{t^{n-1}} \, dt \quad (\text{use } t = |y-x|)$$

$$= \int_0^s \int_{\partial B(x,t)} \frac{|Du(y)|}{|x-y|^{n-1}} \, dS(y) \, dt$$

$$= \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}} \, dy$$

So we have

$$\int_{\partial B(0,1)} |u(x+sw) - u(x)| \, dS(w)$$

$$\leq \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}} \, dy$$

$$\leq \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}} \, dy \quad (\because \ s < r)$$

$\Rightarrow$

$$\underbrace{\int_0^r \int_{\partial B(0,1)} |u(x+sw)-u(x)|\, dS(w)}\, \underline{s^{n-1} ds}$$

$$\leq \int_0^r \underbrace{\left(\int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy\right)}\, \underline{s^{n-1} ds}$$

$$\underset{y=x+sw}{\Longrightarrow} \int_{B(x,r)} |u(y)-u(x)|\, dy$$

$$\leq \left(\int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy\right)\left(\underbrace{\int_0^r s^{n-1} ds}_{\frac{r^n}{n}}\right)$$

$$\Rightarrow \frac{\int_{B(x,r)} |u(y)-u(x)|\, dy}{\alpha_n r^n} \leq \frac{1}{n\alpha_n}\left(\int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy\right)$$

$$\Rightarrow ⨍_{B(x,r)} |u(x)-u(y)|\, dy$$

$$\leq \underbrace{\left(\frac{1}{n\alpha_n}\right)}_{C(n)} \cdot \int_{B(x,r)} \frac{|Du(y)|\, dy}{|x-y|^{n-1}}$$

#.

---

pf of the Morrey's inequality.

First we prove that

$$|u(x)| \leq C(p,n)\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

We have $\quad |u(x)| \leq |u(x)-u(y)| + |u(y)|$

$$\Rightarrow \int_{B(x,1)} |u(x)|\, dy \leq \int_{B(x,1)} |u(x)-u(y)|\, dy + \int_{B(x,1)} |u(y)|\, dy$$

$$|u(x)|\, \underline{|B(x,1)|} \leq \int_{B(x,1)} |u(x)-u(y)|\, dy + \int_{B(x,1)} |u(y)|\, dy$$

$$\Rightarrow |u(x)| \leq ⨍_{B(x,1)} |u(x)-u(y)|\, dy + ⨍_{B(x,1)} |u(y)|\, dy$$

From previous lemma

$$⨍_{B(x,1)} |u(x)-u(y)|\,dy \le C(n)\int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy$$

$$\underset{\substack{\uparrow\\ \text{Hölder ineq}}}{\le} C(n)\cdot\left(\int_{B(x,1)} |Du(y)|^p\,dy\right)^{\frac{1}{p}}\cdot\underline{\left(\int_{B(x,1)}\left(\frac{1}{|x-y|^{n-1}}\right)^{\frac{p}{p-1}}dy\right)^{\frac{p-1}{p}}}$$

$$\left(\frac{1}{p}+\underbrace{\frac{p-1}{p}}_{=\frac{1}{q}}=1\right)$$

$$\int_{B(x,1)} \frac{1}{|x-y|^{(n-1)\cdot\frac{p}{p-1}}}\,dy \underset{\substack{\uparrow\\ r=|x-y|}}{=} \alpha_n\int_0^1 \frac{1}{r^{\frac{(n-1)p}{p-1}}}\, r^{n-1}\,dr$$

$$= \alpha_n\int_0^1 r^{\,n-1-\frac{(n-1)p}{p-1}}\,dr$$

$$= \alpha_n \left.\frac{r^{\,n-\frac{(n-1)p}{p-1}}}{n-\frac{(n-1)p}{p-1}}\right]_0^1 \qquad \left(n-\frac{(n-1)p}{p-1} = \frac{np-n-np+p}{p-1} = \frac{p-n}{p-1} > 0\right)$$

$$= \frac{\alpha_n}{n-\frac{(n-1)p}{p-1}} = \frac{\alpha_n(p-1)}{(p-n)} = C(p,n) \qquad \text{b/c } p>n$$

$$\Rightarrow ⨍_{B(x,1)} |u(x)-u(y)|\,dy$$

$$\le C(n,p)\left(\int_{B(x,1)} |Du(y)|^p\,dy\right)^{\frac{1}{p}}$$

$$\left(⨍_{B(x,1)} |u(y)|\,dy\right) \le C(n,p)\left(\int_{B(x,1)} |u(y)|^p\,dy\right)^{\frac{1}{p}}$$

$(\Longrightarrow)$

$$|u(x)| \leq \fint_{B(x,1)} |u(x) - u(y)|\, dy + \fint_{B(x,1)} |u(y)|\, dy$$

$$\leq C_1(n,p) \left(\int_{B(x,1)} |Du(y)|^p \, dy \right)^{\frac{1}{p}} +$$

$$C_2(n,p) \left(\int_{B(x,1)} |u(y)|^p \, dy \right)^{\frac{1}{p}}$$

$$\leq C(n,p) \, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

(Ctn on the pf of Morrey's inequality)

Last time, we proved that

PDE NOTES
on Nov.19
6 pages

$$|u(x)| \leq C(n,p)\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

$$\sup_{x\in\mathbb{R}^n} |u(x)| \leq C(n,p)\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

Recall we have proved the following lemma.

Lemma: $$\fint_{B(x,r)} |u(x)-u(y)|\, dy \leq C(n) \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy$$

This lemma implies that

$$\fint_{B(x,r)} |u(x)-u(y)|\, dy \leq C(n,p)\, r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

Pf: We have

$$\fint_{B(x,r)} |u(x)-u(y)|\, dy$$

$$\leq C(n) \left(\int_{B(x,r)} |Du(y)| \cdot \frac{1}{|x-y|^{n-1}}\, dy \right)$$

$$\left(\frac{1}{p} + \underbrace{\frac{p-1}{p}}_{=\frac{1}{q}} = 1 \right)$$

Hölder ineq

$$\leq C(n) \left(\int_{B(x,r)} |Du(y)|^p dy \right)^{\frac{1}{p}} \underbrace{\left(\int_{B(x,r)} \left(\frac{1}{|x-y|^{n-1}} \right)^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}}}$$

Hölder ineq

$$\le C(n)\left(\int_{B(x,r)} |Du(y)|^p\,dy\right)^{\frac{1}{p}}\underbrace{\left(\int_{B(x,r)}\left(\frac{1}{|x-y|^{n-1}}\right)^{\frac{p}{p-1}}dy\right)^{\frac{p-1}{p}}}$$

Compute $\displaystyle\int_{B(x,r)} |x-y|^{-(n-1)\frac{p}{p-1}}\,dy$

$$\overset{|x-y|=t}{=} \int_0^r t^{-(n-1)\frac{p}{p-1}}(n\alpha_n)\,t^{n-1}\,dt$$

$$=(n\alpha_n)\int_0^r t^{\,n-1-\frac{(n-1)p}{p-1}}\,dt$$

$$=(n\alpha_n)\left.\frac{t^{\,n-\frac{(n-1)p}{p-1}}}{n-\frac{(n-1)p}{p-1}}\right]_0^r$$

$$\boxed{\begin{aligned} &n-\frac{(n-1)p}{p-1}\\ &=\frac{np-n-np+p}{p-1}\\ &=\frac{p-n}{p-1}>0\\ &\text{b/c } p>n\end{aligned}}$$

$$=n\alpha_n\,\frac{r^{\frac{p-n}{p-1}}}{\frac{p-n}{p-1}}$$

$$=C(n,p)\,r^{\frac{p-n}{p-1}}$$

$$\Rightarrow \fint_{B(x,r)} |u(x)-u(y)|\,dy$$

$$\le C(n,p)\,\|Du\|_{L^p(B(x,r))}\left(r^{\frac{p-n}{p-1}}\right)^{\frac{p-1}{p}}$$

$$=C(n,p)\,\|Du\|_{L^p(B(x,r))}\,r^{1-\frac{n}{p}}$$

Next, we'll use the estimate

$$\text{(*)} \quad \fint_{B(x,r)} |u(x) - u(y)| \, dy \le C(n,p) r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

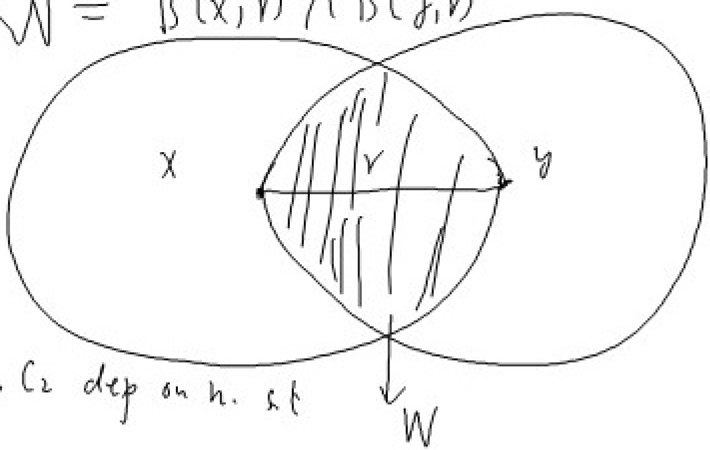
to prove that

$$\sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^{1-\frac{n}{p}}} \le C \|Du\|_{L^p(\mathbb{R}^n)}$$

Choose any two point $x, y \in \mathbb{R}^n$

with $r = |x-y|$

Let $W = B(x,r) \cap B(y,r)$



$\exists$ C_1, C_2 dep on n. s.t

$$C_1 \left(\mathrm{Vol}(B(x,r)) \right) \le \mathrm{Vol}(W) \le C_2 \left(\mathrm{Vol}(B(x,r)) \right)$$

$$|u(x) - u(y)| \le |u(x) - u(z)| + |u(z) - u(y)|$$

$$\int_W |u(x)-u(y)|\,dz \le \int_W \big(|u(x)-u(z)|+|u(z)-u(y)|\big)\,dz$$

$$|u(x)-u(y)|\,\mathrm{Vol}(W) \le \left(\int_W |u(x)-u(z)|\,dz\right) + \left(\int_W |u(z)-u(y)|\,dz\right)$$

$$\Rightarrow |u(x)-u(y)| \le \frac{\int_W |u(x)-u(z)|\,dz}{\mathrm{Vol}(W)} + \frac{\int_W |u(z)-u(y)|\,dz}{\mathrm{Vol}(W)}$$

We have

$$\frac{\int_W |u(x)-u(z)|\,dz}{\mathrm{Vol}(W)} \qquad (W \subset B(x,r))$$

$$\le \frac{C(n)\int_{B(x,r)} |u(x)-u(z)|\,dz}{\mathrm{Vol}(B(x,r))} \qquad \left(\frac{1}{\mathrm{Vol}(W)} \le \frac{1}{\underbrace{C_1 b_0}_{C(n)}|B(x,r)|}\right)$$

$$= C(n) \underbrace{⨍_{B(x,r)} |u(x)-u(z)|\,dz}$$

$$\le C(n)\cdot \underbrace{C(n,p)\, r^{1-\frac{n}{p}}}_{\text{from } *} \|Du\|_{L^p(B(x,r))}$$

$$\le C(n,p)\,|x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

Similarly, we also have

$$\frac{\int_W |u(y)-u(z)|\,dz}{\mathrm{Vol}(W)} \le C(n,p)\,|x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(B(y,r))}$$

$$\Rightarrow |u(x)-u(y)| \le C(n,p)\,|x-y|^{1-\frac{n}{p}} \left(\|Du\|_{L^p(B(x,r))} + \|Du\|_{L^p(B(y,r))}\right)$$

$$\le 2C(n,p)\,|x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \quad \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq C_{(n,p)} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \quad \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq C(n,p) \|Du\|_{L^p(\mathbb{R}^n)}$$

Combining this with

$$\sup_{x \in \mathbb{R}^n} |u(x)| \leq C(n,p) \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |u(x)| + \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}}$$

$$\leq C(n,p) \|Du\|_{L^p(\mathbb{R}^n)}$$

#

Remark:

The Key Lemma

$$\underbrace{\fint_{B(x,r)} |u(x) - u(y)| \, dy}_{\;} \le C(n,p)\, r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

$$\ge$$

$$\left| u(x) - \fint_{B(x,r)} u(y) \right|$$

PDE NOTES
on Nov.21
5 pages

Last time, we proved the Morrey's inequality for C^1 ftn i.e.

$$\|u\|_{C^{0,r}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

when $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$,

$n < p \leq \infty$, $r = 1 - \frac{n}{p}$

Next, we'll use density Th to prove that

Th: Let Ω be a bounded, open subset of $\mathbb{R}^n$ and suppose $\partial\Omega$ is C^1

Assume $n < p \leq \infty$ and $u \in W^{1,p}(\Omega)$

Then $\exists\, u^*$ s.t $u^* = u$ a.e. and

$u^* \in C^{0,r}(\bar{\Omega})$ for $r = 1 - \frac{p}{n}$ with

the estimate $\|u^*\|_{C^{0,r}(\bar{\Omega})} \leq C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$

$C^{0,\gamma}(\overline{\Omega})$ $\qquad W^{1,p}(\Omega)$

pf: Since $\partial\Omega$ is C^1,

by Th 1 on §5.3

we can find an extension $\bar{u} = Eu \in W^{1,p}(\mathbb{R}^n)$ s.t

① $\bar{u} = u$ in Ω

② $\bar{u}$ has compact support in V, where $\Omega \subset\subset V \subset\subset \mathbb{R}^n$

③ $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C(p,\Omega,V)\,\|u\|_{W^{1,p}(\Omega)}$

Since $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ and it has cpt support

$\exists\ \{u_m\}_{m=1}^{\infty}$, $u_m \in C_0^{\infty}(\mathbb{R}^n)$ s.t $supp(u_m) \subset\subset V$

$u_m \to \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$

By Morrey inequality for $C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$

$\Rightarrow$ $\|u_m - u_\ell\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C(n,p,\Omega)\,\|u_m - u_\ell\|_{W^{1,p}(\mathbb{R}^n)}$

b/c $u_m \to \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$

$\Rightarrow$ $\{u_m\}$ is a cauchy seq in $W^{1,p}(\mathbb{R}^n)$

implies that $\{u_m\}$ is a cauchy seq in $C^{0,\gamma}(\mathbb{R}^n)$, $\gamma = 1 - \frac{n}{p}$.

Recall that $C^{0,\gamma}(\mathbb{R}^n)$ is a Banach space

$\Rightarrow$ $u_m \xrightarrow[m\to\infty]{} u^* \in C^{0,\gamma}(\mathbb{R}^n)$

Since $u_m \to \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$, we know

$\bar{u} = u^*$ a.e. $\Rightarrow$ $u = u^*$ a.e in Ω

(b/c $\bar{u} = u$ in Ω)

Also, we have

$$\|u_m\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C(n,p,\Omega)\,\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$$

and $\lim_{m\to\infty} u_m = u^*$ in $C^{0,\gamma}(\mathbb{R}^n)$

and $\lim_{m\to\infty} u_m = \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$

Let $m\to\infty$

$$\Longrightarrow \quad * \quad \|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C(n,p,\Omega)\,\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}$$

Note that

$$** \quad \begin{cases} \|u^*\|_{C^{0,\gamma}(\bar{\Omega})} \le \|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \\ \text{and} \quad \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C_1(n,V,\Omega)\,\|u\|_{W^{1,p}(\Omega)} \end{cases}$$

(V depends on Ω)

From $*$, $**$, we have

$$\|u^*\|_{C^{0,\gamma}(\bar{\Omega})} \le \tilde{C}(n,p,\Omega)\,\|u\|_{W^{1,p}(\Omega)}$$

#

So far, we proved that.

Ω bounded, open subset in $\mathbb{R}^n$ and $\partial\Omega$ is C^1. $u \in W^{1,p}(\Omega)$

Case 1 : $1 \le p < n$

$$\Rightarrow \left| \begin{array}{l} u \in L^q(\Omega) \text{ for } 1 \le q \le p^* = \frac{np}{n-p} \\ \|u\|_{L^q(\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)} \end{array} \right.$$

Case 2 : $n < p \le \infty$

$$\Rightarrow \left\{ \begin{array}{l} u \in C^{0,1-\frac{n}{p}}(\Omega) \text{ and} \\ \|u\|_{C^{0,1-\frac{n}{p}}(\bar{\Omega})} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)} \end{array} \right.$$

We can generalize this result to $u \in W^{k,p}(\Omega)$.

Th (General Sobolev inequalities)

Let Ω be a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary.

Assume $u \in W^{k,p}(\Omega)$

(i) If $kp < n$

then $u \in L^q(\Omega)$ where

$$1 \leq q \leq \frac{pn}{n-kp} \left(= \frac{1}{\frac{1}{p} - \frac{k}{n}} \right).$$

We also have

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}$$

(ii) If $kp > n$

then $u \in C^{k - [\frac{n}{p}] - 1, \gamma}(\bar{\Omega})$, where

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, \text{ if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, \text{ if } \frac{n}{p} \text{ is an integer} \end{cases}$$

We also have the estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar{\Omega})} \leq C(n,p,\Omega) \|u\|_{W^{k,p}(\Omega)}$$

pf: Sketch of the pf:

$$D^\beta u \in W^{1,p}(\Omega)$$

for $|\beta| \leq k-1$

Remark: we have

$$n > kp \Rightarrow W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$$

$$1 \leq q \leq \frac{pn}{n-kp}$$

In fact, we'll prove that $W^{k,p}(\Omega) \subset\subset L^q(\Omega)$

PDE NOTES
on Nov.24
5 pages

§5.7 Compactness

We have proven that

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

if $1 \le p < n$ and $1 \le q \le p^* = \frac{np}{n-p}$

$\begin{cases} \Omega \text{ is bounded} \\ \partial\Omega \text{ is } C^1. \end{cases}$

In the following, we'll prove that:

$$\underset{\text{cpt embedded}}{W^{1,p}(\Omega) \subset\subset L^q(\Omega)}$$

for $1 \le q < p^*$

Def: Let X and Y be Banach spa.

$X \subset Y$, We say X is compactly embedded

in Y, written $X \subset\subset Y$,

provided

① $\|x\|_Y \le C \|x\|_X$ for $x \in X$

and some constant C indep of x

② each bounded sequence in X

is precompact in Y.

(i.e. $\{x_m\}$ is a bounded seq in X

$\exists$ subseq $\{x_{m_j}\}_{j=1}^{\infty}$ s.t. $\{x_{m_j}\}$ converges in Y)

Note that this subsequence may not

converge in X.

Remark: $\overline{X} \subset Y$ if $X \subset\subset Y$

Pf: $y \in \overline{X}$, $\exists\ y_n \to y$ s.t. $\lim_{n\to\infty} \|y_n - y\|_X = 0$

$$\|y_n - y\|_Y \le C \|y_n - y\|_X$$

$\Rightarrow \lim_{n\to\infty} \|y_n - y\|_Y = 0 \Rightarrow y \in Y$

<u>Th 1</u> on p272 (Rellich-Kondrachov Compactness Theorem)

Assume Ω is a bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1. Suppose $1 \le p < n$. Then

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \le q < p^*$.

Pf: We have proved that

$$\|u\|_{L^q(\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$$

for $1 \le q \le p^*$.

To prove that $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for $1 \le q < p^*$, it suffices to show that if $\{u_m\}_{m=1}^{\infty}$ is a bounded seq in $W^{1,p}(\Omega)$

$\|u_m\|_{W^{1,p}(\Omega)} \le C$, $\exists$ a subseq $\{u_{m_j}\}$ which converges in $L^q(\Omega)$.

Let us prove the following Lemma first.

Lemma 1: if $u \in C^1(V) \cap W^{1,p}(V)$ and $\operatorname{supp} u \subset\subset V$ where V is bounded.

$\Longrightarrow$ $\|u-u^\varepsilon\|_{L^1(V)} \le \varepsilon\, C(n,p,V)\, \|Du\|_{L^p(\Omega)}$

for ε small enough.

pf: Recall that $u^\varepsilon = \eta_\varepsilon * u$.

$$u^\varepsilon(x) = \int_{B(0,1)} \eta(y)\, u(x-\varepsilon y)\, dy,$$

$$\int_{B(0,1)} \eta(y)\, dy = 1,$$

$$u(x) = \int_{B(0,1)} \eta(y)\, u(x)\, dy$$

$$\left(\begin{array}{ll} \text{b/c} & \operatorname{supp} u \subset\subset V \\ \Rightarrow & \operatorname{supp} u^\varepsilon \subset\subset V \text{ if } \varepsilon \text{ is small enough.} \end{array} \right)$$

$$|u^\varepsilon(x) - u(x)| = \left| \int_{B(0,1)} \eta(y) \underbrace{\left(u(x-\varepsilon y) - u(x)\right)} dy \right|$$

$$= \left| \int_{B(0,1)} \eta(y) \left(\int_0^1 \left(\frac{d}{dt} u(x-\varepsilon t y)\right) dt \right) dy. \right|$$

$\shortparallel$

$$= \left| \int_{B(0,1)} \eta(y) \left(\int_0^1 \left(\frac{d}{dt} u(x-\varepsilon t y) \right) dt \right] dy \right|$$

$$\| \qquad \left(\int_0^1 Du(x-\varepsilon t y)\cdot(-\varepsilon y)\, dt \right]$$

b/c $\|y\| \le 1$

$$\le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 |Du(x-\varepsilon t y)| dt\, dy$$

$$\|u-u^\varepsilon\|_{L^1(V)} = \int_V |u(x)-u^\varepsilon(x)|\, dx$$

$$\le \varepsilon \int_V \int_{B(0,1)} \eta(y) \int_0^1 |Du(x-\varepsilon t y)| dt\, dy\, dx$$

$$= \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \underbrace{\int_V \|Du(x-\varepsilon t y)\| dx}_{= \|Du\|_{L^1(V)} \quad z = x-\varepsilon t y} dt\, dy$$

$$= \varepsilon \|Du\|_{L^1(V)} \underbrace{\int_{B(0,1)} \eta(y)\, dy}_{=1}$$

$$= \varepsilon \int_V |Du(x)|\, dx$$

Hölder ineq

$$\le \varepsilon \left(\int_V |Du|^p dx \right)^{\frac{1}{p}} \left(\int_V (1)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}}$$

$\frac{1}{p} + \frac{p-1}{p} = 1$

$$= \varepsilon \|Du\|_{L^p(V)} \underbrace{(\mathrm{Vol}(V))^{\frac{p-1}{p}}}_{= C(n,p,V)}$$

$$\Rightarrow \|u-u^\varepsilon\|_{L^1(V)} \le \varepsilon C(n,p,V) \|Du\|_{L^p(V)}$$

<u>Cor</u> $u \in W^{1,p}(V)$, $\operatorname{supp} u \subset\subset V$

$$\Rightarrow \|u-u^\varepsilon\|_{L^1(V)} \le \varepsilon C(n,p,V) \|Du\|_{L^p(V)}$$

pf) $u \in W^{1,p}(V)$, $\operatorname{supp} u \subset\subset V$

$\exists$ $u_m \in C_0^\infty(V)$ s.t.

$u_m \to u$ in $W^{1,p}(\Omega)$

i.e. $u_m \to u$ in $L^p(V)$

and $Du_m \to Du$ in $L^p(V)$

b/c $u_m \in C_0^\infty(V)$

From previous Lemma,

(*) $\|u_m - u_m^\varepsilon\|_{L^1(V)} \le \varepsilon\, C(V,n,p)\, \|Du_m\|_{L^p(V)}$

Note that $u_m^\varepsilon \xrightarrow[m\to\infty]{} u^\varepsilon$ in $L_1(V)$

$$\Bigg(\int_V |u_m^\varepsilon(x) - u^\varepsilon(x)|\, dx$$

$$\le \int_V \int_{B(0,1)} \eta(y)\, |u_m(x-\varepsilon y) - u(x-\varepsilon y)|\, dy\, dx$$

$$\le \int_V |u_m(x-\varepsilon y) - u(x-\varepsilon y)|\, dx$$

$$= \|u_m - u\|_{L^1(V)}$$

$$\le C(n,p,V)\, \|u_m - u\|_{L^p(V)}$$

Hölder ineq.

b/c $\|u_m - u\|_{L^p(V)} \to 0$

$\Rightarrow \|u_m^\varepsilon - u^\varepsilon\|_{L^1(V)} \to 0$ as $m \to \infty$

From *, take $m \to \infty$

$\Rightarrow \|u - u^\varepsilon\|_{L^1(V)} \le \varepsilon\, C(n,p,V)\, \|Du\|_{L^p(V)}$

PDE NOTES
on Dec.1
3 pages

Continuation from Last time,

Recall that we have proved

Cor1 : $u \in W^{1,p}(V)$, $supp(u) \subset\subset V \subset\subset \mathbb{R}^n$

$\Rightarrow$ $\|u - u^{\varepsilon}\|_{L^1(V)} \leq \varepsilon\, C(V,n,p) \|Du\|_{L^p(V)}$

for ε small enough.

(So $supp(u^{\varepsilon}) \subset\subset V$)

Now, we'll prove the following Lemma:

Lemma 2 : Suppose $\{u_m\}_{m=1}^{\infty}$ where $supp(u_m) \subset\subset V \subset\subset \mathbb{R}^n$ and

$\exists$ C s.t. $\|u_m\|_{L^1(V)} < C$ for all m.

$\Longrightarrow$ For each $\varepsilon > 0$ small enough

the seq $\{u_m^{\varepsilon}\}_{m=1}^{\infty}$ is uniformly bounded

and equi-continuous

Remark: Under the same condition,

b/c $\{u_m^{\varepsilon}\}_{m=1}^{\infty}$ is uniformly bounded and

equi-continuous. Also $u_m^{\varepsilon} \in C_0^{\infty}(V)$ for

ε small enough.

$\Rightarrow$ By Arzela-Ascoli Th

For each ε small enough $\{u_m^{\varepsilon}\}_{m=1}^{\infty}$ has

a convergent subsequence $\{u_{m_j}^{\varepsilon}\}_{j=1}^{\infty}$ in $C^0(\bar{V})$

$\Rightarrow$ b/c $\bar{V}$ is compact

So $\{u_{m_j}^{\varepsilon}\}_{j=1}^{\infty}$ also converge in $L^q(V)$

for $1 \leq q \leq \infty$.

pf:

Recall $u_m^\varepsilon(x) = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, u_m(y)\, dy.$

$$= \int_{B(x,\varepsilon)} \left(\frac{\eta\left(\frac{x-y}{\varepsilon}\right)}{\varepsilon^n} \right) u_m(y)\, dy$$

$$\Rightarrow \quad |u_m^\varepsilon(x)| \le \frac{\|\eta\|_\infty}{\varepsilon^n} \int_{B(x,\varepsilon)} |u_m(y)|\, dy$$

$$\le \frac{\|\eta\|_\infty}{\varepsilon^n} \|u_m\|_{L^1(V)}$$

$$\le \frac{C \|\eta\|_\infty}{\varepsilon^n} \quad \left(\text{b/c } \|u_m\|_{L^1(V)} < C \right)$$

$$\Rightarrow \quad \|u_m^\varepsilon\|_\infty \le \frac{C\|\eta\|_\infty}{\varepsilon^n}$$

$\Rightarrow$ $\{u_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded for each $\varepsilon > 0$ (small enough)

To show that $\{u_m^\varepsilon\}_{m=1}^\infty$ is equicontinuous,

it suffices to prove that $\{Du_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded.

$$Du_m^\varepsilon(x) = \int_{B(x,\varepsilon)} D\eta_\varepsilon(x-y)\, u_m(y)\, dy \quad \left(\eta_\varepsilon(x) = \frac{\eta\left(\frac{x}{\varepsilon}\right)}{\varepsilon^n},\ D\eta_\varepsilon = \frac{(D\eta)\left(\frac{x}{\varepsilon}\right)}{\varepsilon^{n+1}} \right)$$

$$= \frac{1}{\varepsilon^{n+1}} \int_{B(x,\varepsilon)} (D\eta)\left(\frac{x-y}{\varepsilon}\right) u_m(y)\, dy$$

$$\|Du_m^\varepsilon\|_\infty \le \frac{\|D\eta\|_\infty}{\varepsilon^{n+1}} \int_{B(x,\varepsilon)} |u_m(y)|\, dy$$

$$\le \frac{\|D\eta\|_\infty}{\varepsilon^{n+1}} \|u_m\|_{L^1(V)} \le \frac{C\|D\eta\|_\infty}{\varepsilon^{n+1}}$$

$\Rightarrow$ $\{Du_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded for $\varepsilon > 0$ and small enough.

(pf of the compactness Th)

$\{u_m\}_{m=1}^{\infty}$ is a bounded sequence in $W^{1,p}(\Omega)$

Slect $V \subset\subset \mathbb{R}^n$ st $\Omega \subset\subset V \subset\subset \mathbb{R}^n$.

Let $\{\bar{u}_m\}_{m=1}^{\infty}$ be the extension of u_m st

(i) $\operatorname{supp}(\bar{u}_m) \subset\subset V$ (iii) $\bar{u}_m = u_m$ in Ω

(ii) $\bar{u}_m \in W^{1,p}(\mathbb{R}^n)$

$$\|\bar{u}_m\|_{W^{1,p}(\mathbb{R}^n)} \leq C(p,\Omega)\|u_m\|_{W^{1,p}(\Omega)}$$

b/c $\{u_m\}_{m=1}^{\infty}$ is a bounded seq in $W^{1,p}(\Omega)$

$\Rightarrow$ $\{\bar{u}_m\}_{m=1}^{\infty}$ is also a bounded seq in $W^{1,p}(\mathbb{R}^n)$

To prove compactness Th, it suffices

to show that $\{\bar{u}_m\}_{m=1}^{\infty}$ has a convergent subseq

$\{\bar{u}_{m_j}\}_{j=1}^{\infty}$ in $L^q(\mathbb{R}^n)$ for $1 \leq q < p^*$.

(b/c $\|\bar{u}_{m_j} - \bar{u}\|_{L^q(\Omega)} \leq \|\bar{u}_{m_j} - \bar{u}\|_{L^q(\mathbb{R}^n)}$.

$\|$

$\|u_{m_j} - \bar{u}\|_{L^q(\Omega)}$

$\because$ $\bar{u}_{m_j} = u_{m_j}$ in Ω)

12/03/08

PDE NOTES
on Dec.3
4 pages

Now we have $\{\bar{u}_m\}_{m=1}^{\infty}$ where

$$\sup_m \|\bar{u}_m\|_{W^{1,p}(V)} \leq C \,, \quad \mathrm{supp}(\bar{u}_m) \subset\subset V$$

Consider $\bar{u}_m^{\varepsilon} = \eta_{\varepsilon} * \bar{u}_m$.

By Cor1, we have

$$\begin{aligned} \|\bar{u}_m - \bar{u}_m^{\varepsilon}\|_{L^1(V)} &\leq \varepsilon C \|D\bar{u}_m\|_{L^p(V)} \\ &\leq \varepsilon C \underbrace{\|\bar{u}_m\|_{W^{1,p}(V)}}_{\text{uniformly bounded}} \\ &\leq \varepsilon C \leftarrow \text{indep of } m. \end{aligned}$$

$$\Rightarrow \lim_{\varepsilon \to 0} \|\bar{u}_m - \bar{u}_m^{\varepsilon}\|_{L^1(V)} = 0 \quad (\text{indep of } m)$$

Recall that the interpolation inequality

$$1 \leq s \leq r \leq t, \quad \frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$$

$$\|f\|_{L^r(V)} \leq \|f\|_{L^s(V)}^{\theta} \|f\|_{L^t(V)}^{1-\theta}$$

Now $\quad 1 \leq q < p^*$

$$\frac{1}{p^*} < \frac{1}{q} \leq 1$$

$$\frac{1}{q} = \theta \cdot 1 + \frac{(1-\theta)}{p^*}$$

$\theta \qquad 1-\theta$

$$\Rightarrow \quad \|\bar{u}_m^\varepsilon - \bar{u}_m\|_{L^q(V)} \le \|\bar{u}_m^\varepsilon - \bar{u}_m\|_{L^1(V)}^\theta \cdot \|\bar{u}_m^\varepsilon - \bar{u}_m\|_{L^{p^*}(V)}^{1-\theta}$$

(By interpolation ineq.)

Recall that $\bar{u}_m \in W^{1,p}(V)$, $supp(\bar{u}_m) \subset\subset V$

$$\|\bar{u}_m\|_{L^{p^*}(V)} \le C \|\bar{u}_m\|_{W^{1,p}(V)}$$

$$\|\bar{u}_m^\varepsilon\|_{L^{p^*}(V)} \le \|\bar{u}_m\|_{L^{p^*}(V)}$$

(Exercise)

$$\Rightarrow \quad \|\bar{u}_m - \bar{u}_m^\varepsilon\|_{L^{p^*}(V)} \le \|\bar{u}_m\|_{L^{p^*}(V)} + \|\bar{u}_m^\varepsilon\|_{L^{p^*}(V)}$$

$$\le 2 \|\bar{u}_m\|_{L^{p^*}(V)}$$

$$\le 2C \|\bar{u}_m\|_{W^{1,p}(V)}$$

$$\le C$$

From $\Rightarrow \quad \|\bar{u}_m^\varepsilon - \bar{u}_m\|_{L^q(V)} \le \|\bar{u}_m^\varepsilon - \bar{u}_m\|_{L^1(V)}^\theta C^{1-\theta}$

Since $\lim_{\varepsilon \to 0} \|\bar{u}_m^\varepsilon - \bar{u}_m\|_{L^1(V)} = 0$ (uniformly in m)

$$\Rightarrow \quad \lim_{\varepsilon \to 0} \|\bar{u}_m^\varepsilon - \bar{u}_m\|_{L^q(V)} = 0 \text{ (uniformly in } m)$$

for $1 \le q < p^*$

Since $supp(\bar{u}_m) \subset\subset V$, $\sup_m \|\bar{u}_m\|_{L^1(V)} < \infty$

(b/c $\int_V |\bar{u}_m| \le \left(\int_V (\bar{u}_m)^p\right)^{\frac{1}{p}} \cdot (Vol(V))^{\frac{p-1}{p}}$

$\le C \cdot \|\bar{u}_m\|_{W^{1,p}(V)}$)

and $supp(\bar{u}_m^\varepsilon) \subset\subset V$ for ε small enough

$\Rightarrow$ By Lemma 2,

For each $\varepsilon > 0$, $\{\bar{u}_m^\varepsilon\}_{m=1}^\infty$ has a convergent subsequence in $L^q(V)$.

Now, fix $\delta > 0$,

We'll show that $\exists$ a subseq

$\{\bar{u}_{m_j}\}_{j=1}^{\infty}$, s.t $j, k \geq N$

$$\Rightarrow \quad \|\bar{u}_{m_j} - \bar{u}_{m_k}\|_{L^q(V)} \leq \delta$$

Recall that $\lim_{\varepsilon \to 0} \|\bar{u}_m^{\varepsilon} - \bar{u}_m\|_{L^q(V)} = 0$

We can find ε small enough s.t

$$\|\bar{u}_m^{\varepsilon} - \bar{u}_m\| < \frac{\delta}{3} \text{ for all } m$$

Now for this ε, we can find a subseq

$\{\bar{u}_{m_j}^{\varepsilon}\}_{j=1}^{\infty}$ in $L^q(V)$ s.t

$$\|\bar{u}_{m_j}^{\varepsilon} - \bar{u}_{m_k}^{\varepsilon}\|_{L^q(V)} < \frac{\delta}{3} \text{ for } j, k \geq N$$

$$\Rightarrow \|\bar{u}_{m_j} - \bar{u}_{m_k}\|_{L^q(V)} \leq \|\bar{u}_{m_j} - \bar{u}_{m_j}^{\varepsilon}\|_{L^q(V)} + \|\bar{u}_{m_j}^{\varepsilon} - \bar{u}_{m_k}^{\varepsilon}\|_{L^q(V)} + \|\bar{u}_{m_k}^{\varepsilon} - \bar{u}_{m_k}\|_{L^q(V)}$$

$$< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta$$

Now let $\delta = 1, \frac{1}{2}, \cdots$ $(\delta \to 0)$

One can use "diagonal process" to find a subseq s.t. $\lim_{j,k\to\infty} \|\bar{u}_{m_j} - \bar{u}_{m_k}\|_{L^q(V)} = 0$

b/c $L^q(V)$ is complete

$\Rightarrow$ $\{\bar{u}_{m_j}\}$ converges in $L^q(V)$ #.

We have proved that

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

where $\begin{cases} 1 \le p < n \\ 1 \le q < p^* = \frac{np}{n-p} \ (> p) \\ \Omega \text{ is bounded}, \partial\Omega \text{ is smooth.} \end{cases}$

Remark:

1° $W_0^{1,p}(\Omega) \subset\subset L^q(\Omega)$

even if $\partial\Omega$ is not C^1.

(Homework.)

2° $p \to n^-$, $p^* \to \infty$

$W^{1,p}(\Omega) \subset\subset L^p(\Omega)$.

(Discuss next time!)

$p^* > p$

$$\left(\text{b/c. } \frac{np}{n-p} - p = \frac{np - np + p^2}{n-p} = \frac{p^2}{n-p} > 0 \right)$$

PDE NOTES
on Dec.5
6 pages

① Recall that $\Omega \subset\subset \mathbb{R}^n$, $1 \le p < n$

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for $1 \le q < p^* = \frac{np}{n-p}$

b/c $p^* - p = \frac{np}{n-p} - p = \frac{np - np + p^2}{n-p} = \frac{p^2}{n-p} > 0$

$\Rightarrow$ $1 \le p < p^*$

$\Rightarrow$ $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$

when $1 \le p < n$.

② When $p = n$

b/c Ω is bounded

$W^{1,n}(\Omega) \subset W^{1,r}(\Omega)$ for $1 \le r < n$

by Hölder inequality

(b/c $L^s(\Omega) \subset L^t(\Omega)$ if $1 \le t \le s$)

Ω is bounded.

$r^* = \frac{rp}{p-r}$ $\Rightarrow$ $r^* \to \infty$ as $r \to p^-$

$$W^{1,n}(\Omega) \subset W^{1,r}(\Omega) \subset\subset L^q(\Omega)$$

$1 \le q < r^*$

We can find $r < p$ and close to p

s.t $n < r^*$. Choose $q = n$

$\Rightarrow$ $W^{1,n}(\Omega) \subset W^{1,r}(\Omega) \subset\subset L^n(\Omega)$

for this choice of r.

$\Rightarrow$ $W^{1,n}(\Omega) \subset\subset L^n(\Omega)$

when $p = n$.

(3) $n < p \le \infty$

By Morrey's inequality, we have

$$\|u\|_{C^{0,\gamma}(\bar\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$$

$$\| \quad \|u\|_{C^0(\bar\Omega)} + \sup_{\substack{x\ne y \\ x,y\in\Omega}} \frac{|u(x)-u(y)|}{\|x-y\|^\gamma}$$

where $\gamma = 1 - \frac{n}{p}$ $(0 < \gamma \le 1)$

Claim: $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ when $n < p \le \infty$

From Sobolev inequality, we know

$$\underbrace{W^{1,p}(\Omega) \subset C^{0,\gamma}(\bar\Omega)} \subset L^p(\Omega)$$

To show $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$,

it suffices to prove that

if $\{u_n\}_{n=1}^\infty$ is bounded in $W^{1,p}(\Omega)$

$\Rightarrow \exists$ subsequence $\{u_{n_j}\}_{n_j=1}^\infty$ s.t. it converges in $L^p(\Omega)$

By Sobolev ineq,

$$\Rightarrow \quad \|u_n\|_{C^0(\bar\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|u_n(x)-u_n(y)|}{\|x-y\|^\gamma} < C$$

$\Rightarrow \{u_n\}$ is uniformly bounded

$0<\gamma<1$

$$|u_n(x)-u_n(y)| < C\|x-y\|^\gamma \quad (x\ne y)$$

$\Rightarrow \{u_n\}$ is equi-continuous

By Arzela-Ascoli Th

$\{u_n\}$ has a convergent subsequence $\{u_{n_j}\}_{n_j=1}^{\infty}$ in $C^0(\bar{\Omega})$

So this subsequence also converges in $L^p(\Omega)$. #.

Remark: A bounded set in $C^{k,r}(\bar{\Omega})$ is precompact.

So we have $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ ~~for~~ all p.

$(\Rightarrow \|u\|_{L^p(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)})$

Notation: $(u)_\Omega = \dfrac{\int_\Omega u}{\text{Vol}(\Omega)}$ = average value of u on Ω

Th (Poincare's inequality)

Let Ω be a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary. Ω is connected

Assume $1 \le p \le \infty$

Then $\exists\ C = C(n,p,\Omega)$ s.t

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \le C\,\|Du\|_{L^p(\Omega)}$$

for each $u \in W^{1,p}(\Omega)$

Remark. This bound only depends on $\|Du\|_{L^p(\Omega)}$.

Pf: We'll prove this by contradiction

If the statement (inequality) were false,

$\Rightarrow \exists \{u_k\}_{k=1}^{\infty}$ s.t

$$\begin{cases} u_k \in W^{1,p}(\Omega) \text{ and} \\ \|u_k - (u_k)_\Omega\|_{L^p(\Omega)} > k \, \|Du_k\|_{L^p(\Omega)} \end{cases}$$

We normalize by defining

$$v_k := \frac{u_k - (u_k)_\Omega}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}}$$

Then $\|v_k\|_{L^p(\Omega)} = 1$, $(v_k)_\Omega = 0$

$$Dv_k = \frac{Du_k}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}}$$

$$\|Dv_k\|_{L^p(\Omega)} = \frac{\|Du_k\|_{L^p(\Omega)}}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}} < \frac{1}{k}$$

$\Rightarrow$ $\|v_k\|_{W^{1,p}(\Omega)} = \|v_k\|_{L^p(\Omega)} + \sum_{|\alpha|=1} \|D^\alpha v_k\|_{L^p(\Omega)}$

$\Rightarrow$ $\{v_k\}_{k=1}^{\infty}$ is a bounded seq in $W^{1,p}(\Omega)$

Recall that $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$

$\Rightarrow \exists$ subsequence $\{v_{k_j}\}_{k_j=1}^{\infty}$ s.t $v_{k_j} \to v$ in $L^p(\Omega)$.

Since $\|v_{k_j}\|_{L^p} = 1$ and $(v_{k_j})_\Omega = 0$

$\Rightarrow$ $\|v\|_{L^p} = 1$ and $(v)_\Omega = 0$

Recall that $\|DV_{k_j}\| < \frac{1}{k_j} \to 0$ as $k_j \to \infty$

and $\|V_{k_j} - V\|_{L^1(\Omega)} \to 0$

(b/c $1 \le p$ and Ω is bounded)

$\phi \in C_0^\infty(\Omega)$ $\quad \|V_{k_j} - V\|_{L^p(\Omega)} \to 0$

$$\Rightarrow \int_\Omega V \frac{\partial \phi}{\partial x_n} dx = \lim_{k_j \to \infty} \int_\Omega V_{k_j} \left(\frac{\partial \phi}{\partial x_n}\right)$$

$$= \lim_{k_j \to \infty} (-1) \int_\Omega \phi \underbrace{\frac{\partial}{\partial x_n} V_{k_j}}_{\text{weak derivative}}$$

b/c $V_{k_j} \in W^{1,p}(\Omega)$

$$= 0 \quad \left(\text{b/c } \left\|\frac{\partial V_{k_j}}{\partial x_n}\right\|_{L^p(\Omega)} \to 0 \Rightarrow \left\|\frac{\partial V_{k_j}}{\partial x_n}\right\|_{L^1(\Omega)} \to 0 \right)$$

$$\Rightarrow \underbrace{\frac{\partial V}{\partial x_n}}_{\text{weak derivative}} = 0$$

$\Rightarrow DV = 0$. b/c Ω is connected

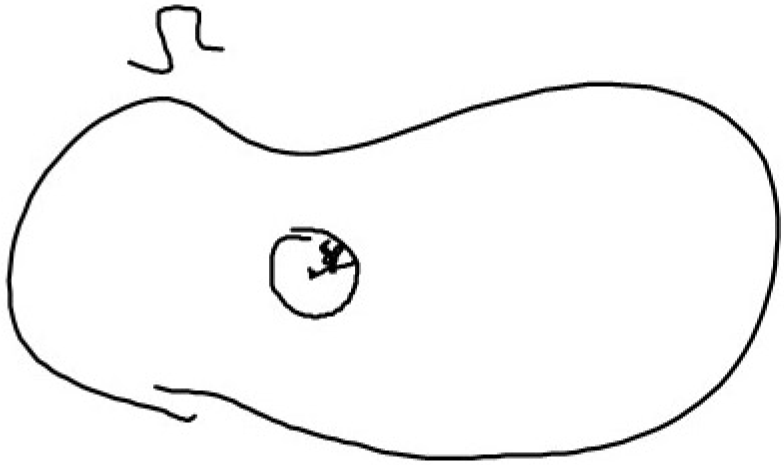
$\Rightarrow v$ is constant a.e.

Since $(v)_\Omega = 0 \Rightarrow v \equiv 0$ a.e.

This contradicts with the fact that

$$\|v\|_{L^p(\Omega)} = 1.$$

Remark: $Du = 0 \quad u \in W^{1,p}(\Omega)$



$$D(\underbrace{u^\varepsilon}_{\text{smooth}}) = (Du)^\varepsilon \quad \text{on } B(x,r) \subset\subset \Omega$$

$$= 0 \text{ if } Du = 0$$

$\Rightarrow u^\varepsilon = C(\varepsilon)$ constant

$$u^\varepsilon \xrightarrow[\text{a.e.}]{} u \quad \text{as } \varepsilon \to 0$$

$\Rightarrow \quad u \equiv$ const a.e. in $B(x,r)$

$\because \Omega$ is connected $\Rightarrow u \equiv$ const a.e.

PDE NOTES
on Dec.8
4 pages

Last time, we proved the Poincare's inequality

Th Let $\Omega \subset\subset \mathbb{R}^n$ with C^1 boundary.
open, connected

Assume $1 \le p < \infty$

Then $\exists$ a constant $C(n,p,\Omega)$ s.t

$$\| u - (u)_\Omega \|_{L^p(\Omega)} \le C \| Du \|_{L^p(\Omega)}$$

for each $u \in W^{1,p}(\Omega)$.

then $(u)_\Omega = \dfrac{\int_\Omega u(y) dy}{\mathrm{Vol}(\Omega)}$.

We'll use this to prove the following.

Th2 (Poincare's inequality for a ball).

Assume $1 \le p < \infty$

Then there exists a constant $C(n,p)$ s.t

$$\| u - (u)_{B(x,r)} \|_{L^p(B(x,r))} \le \underbrace{C(n,p)\, r}\, \| Du \|_{L^p(B(x,r))}$$

for $u \in W^{1,p}(B(x,r))$

Remark: From previous Th, we have

$$\| u - (u)_{B(x,r)} \|_{L^p(B(x,r))} \le \underset{\substack{\| \\ C(n,p)\, r \,\leftarrow \text{ From this Th.}}}{\widetilde{C}(n,p,B(x,r))} \| Du \|_{L^p(B(x,r))}$$

Pf: $u \in W^{1,p}(B(x,r))$

$C(n,p)\, r \leftarrow$ from this Th.

pf: $u \in W^{1,p}(B(0,1))$

By previous Th,

$$\|u - (u)_{B(0,1)}\|_{L^p(B(0,1))} \le \underset{\overset{\|}{\tilde{C}(n,p)}}{C(n,p,B(0,1))} \|Du\|_{L^p(B(0,1))}$$

Suppose $u \in W^{1,p}(B(x,r))$.

Let $v(y) = u(x+ry)$.

Then $v \in W^{1,p}(B(0,1))$

$$\Rightarrow \|v - (v)_{B(0,1)}\|_{L^p(B(0,1))} \le \tilde{C}(n,p) \|Dv\|_{L^p(B(0,1))}$$

$$\|Dv\|_{L^p(B(0,1))} = \sum_{|\alpha|=1} \|D^\alpha v\|_{L^p(B(0,1))}$$

$$= \sum_{|\alpha|=1} \left(\int_{B(0,1)} |D^\alpha v(y)|^p \, dy \right)^{\frac{1}{p}}$$

$$D^\alpha v(y) = D_y^\alpha (u(x+ry)) \qquad \left(\frac{\partial}{\partial y_j} u(x+ry) = \frac{\partial u(x+ry)}{\partial z_j} \cdot r \right)$$

$$\underset{|\alpha|=1}{=} r (D_x^\alpha u)(x+ry)$$

Let $z = x + ry$

$$\Rightarrow z \in B(x,r), \; dz = r^n \, dy$$

$$= \sum_{|\alpha|=1} \left(\int_{B(x,r)} (r\, |D^\alpha u(z)|)^p r^{-n} \, dz \right)^{\frac{1}{p}}$$

$$= \sum_{|\alpha|=1} r\, r^{-\frac{n}{p}} \left(\int_{B(x,r)} |D^\alpha u(z)|^p \, dz \right)^{\frac{1}{p}}$$

$$= r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

$$(v)_{B(0,1)} = \frac{\int_{B(0,1)} u(x+ry)\, dy}{Vol(B(0,1))} = \frac{\int_{B(x,r)} u(z) r^{-n} dz}{Vol(B(0,1))}$$

$$= \frac{\int_{B(x,r)} u(z)\, dz}{r^n\, Vol(B(0,1))} = (u)_{B(x,r)}$$

$$\|v-(v)_{B(0,1)}\|_{L^p(B(0,1))}$$

$$= \left(\int_{B(0,1)} |u(x+ry)-(u)_{B(x,r)}|^p \, dy\right)^{\frac{1}{p}}$$

$z = x+ry, \quad dz = r^n dy$

$$= \left(\int_{B(x,r)} |u(z)-(u)_{B(x,r)}|^p \, r^{-n} dz\right)^{\frac{1}{p}}$$

$$= r^{-\frac{n}{p}} \|u-(u)_{B(x,r)}\|_{L^p(B(x,r))}.$$

So we have $\|Dv\|_{L^p(B(0,1))} = r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$

and $\|v-(v)_{B(0,1)}\|_{L^p(B(0,1))} = r^{-\frac{n}{p}} \|u-(u)_{B(x,r)}\|_{L^p(B(x,r))}$

From $\|v-(v)_{B(0,1)}\|_{L^p(B(0,1))} \le \tilde{C}(n,p) \|Dv\|_{L^p(B(0,1))}$

$$\Rightarrow \quad r^{-\frac{n}{p}} \|u-(u)_{B(x,r)}\|_{L^p(B(x,r))} \le \tilde{C}(n,p) \, r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

$$\Rightarrow \quad \|u-(u)_{B(x,r)}\|_{L^p(B(x,r))} \le \tilde{C}(n,p) \, r \, \|Du\|_{L^p(B(x,r))}$$

$BMO(\mathbb{R}^n)$ = The space of bounded mean oscillation

$$= \left\{ u \,\middle|\, u \in L^1_{loc}(\mathbb{R}^n),\ \sup_{\substack{B(x,r) \\ \subset \mathbb{R}^n}} \fint_{B(x,r)} |u - (u)_{B(x,r)}|\, dy < \infty \right\}$$

Claim: $W^{1,n}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$

pf: $u \in W^{1,n}(\mathbb{R}^n)$

$$\Rightarrow u \in W^{1,n}(\underline{\underline{B(x,r)}}) \subset W^{1,1}(B(x,r))$$

By previous Th, $\|u - (u)_{B(x,r)}\|_{L^1(B(x,r))} \le C r \|Du\|_{L^1(B(x,r))}$
($p=1$)

$$\le Cr \left(\int_{B(x,r)} |Du(y)|\, dy \right)$$

$$\le C r \left(\int_{B(x,r)} |Du(y)|^n\, dy \right)^{\frac{1}{n}} \cdot \underbrace{\left(\int_{B(x,r)} 1 \right)^{\frac{n-1}{n}}}_{(\alpha_n)^{\frac{n-1}{n}} r^{n-1}}$$

$$\le \tilde{C}\, r^n \|Du\|_{L^n(B(x,r))}$$

$$\Rightarrow \frac{\|u - (u)_{B(x,r)}\|_{L^1(B(x,r))}}{\mathrm{Vol}(B(x,r))} \le \hat{C}(n,p) \|Du\|_{L^n(B(x,r))}$$

$$\fint_{B(x,r)} |u - (u)_{B(x,r)}| \le \hat{C}(n,p) \|Du\|_{L^n(\mathbb{R}^n)}$$

$$\le \hat{C}(n,p) \underline{\underline{\|u\|}}_{W^{1,n}(\mathbb{R}^n)}$$

$$< \infty$$

$u \in W^{1,n}(\mathbb{R}^n)$

#

§ 5.8.2 Difference quotients

PDE NOTES
on Dec.10, 12
5 pages

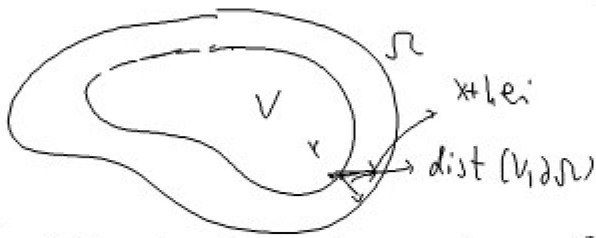
We'll use difference quotient to approximate weak derivatives later on.

Here we study the difference quotients $u : \Omega \longrightarrow \mathbb{R}$

Def: (i) The i-th difference quotient of size h is

$$D_i^h u(x) := \frac{u(x+he_i) - u(x)}{h} \quad (i = 1, \cdots, n)$$

for $x \in V$ and $0 < |h| < \operatorname{dist}(V, \partial\Omega)$



$$(x \in V, \ |h| < \operatorname{dist}(V, \partial\Omega) \Rightarrow x + he_i \in \Omega)$$

(ii) $D^h u := (D_1^h u, D_2^h u, \cdots, D_n^h u)$.

Th3 (Difference quotients and weak derivatives)

(i) Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$.

Then for each $V \subset\subset \Omega$

$$\Rightarrow \quad \|D^h u\|_{L^p(V)} \leq C \|Du\|_{L^p(\Omega)}$$

for some constant $C(n,p)$ and

$$0 < |h| < \frac{1}{2} \operatorname{dist}(V, \partial\Omega)$$

(ii) Assume $1 < p < \infty$, $u \in L^p(V)$ and there exists a constant C such that

$$\|D^h u\|_{L^p(V)} \le C \quad (\text{indep of } h)$$

for all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial\Omega)$

Then $u \in W^{1,p}(V)$ with $\|Du\|_{L^p(V)} \le C$

pf: 1° Assume u is smooth first.

$$u(x+he_i) - u(x) = \int_0^1 \left(\frac{d}{dt} u(x+the_i) \right) dt$$

$$= \int_0^1 (\nabla u(x+the_i) \cdot he_i)\, dt, \quad \|e_i\| = 1$$

$$\Rightarrow |u(x+he_i) - u(x)| \le |h| \int_0^1 \|Du(x+the_i)\|\, dt$$

$$\left| \frac{u(x+he_i) - u(x)}{h} \right| \le \int_0^1 \|Du(x+the_i)\|\, dt$$

$$\Downarrow \quad \left(0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial\Omega) \right)$$

$$|D_i^h u(x)| \le \int_0^1 \|Du(x+the_i)\|\, dt$$

$$\le \left(\int_0^1 \|Du(x+the_i)\|^p\, dt \right)^{\frac{1}{p}} \left(\int_0^1 1^{\frac{p}{p-1}}\, dt \right)^{\frac{p-1}{p}}$$

$$\le \left(\int_0^1 \|Du(x+the_i)\|^p\, dt \right)^{\frac{1}{p}}$$

$$\Rightarrow |D_i^h u(x)|^p \le \int_0^1 \|Du(x+the_i)\|^p\, dt$$

$$\Rightarrow \int_V |D_i^h u(x)|^p\, dx \le \int_V \int_0^1 \|Du(x+the_i)\|^p\, dt\, dx$$

$$= \int_0^1 \int_V \|Du(x+the_i)\|^p\, dx\, dt$$

$$\le \int_0^1 \underline{\int_\Omega \|Du(y)\|^p\, dy}\, dt \quad \left(\begin{matrix} y \\ \| \\ x+the_i \\ \in \Omega \end{matrix} \right)$$

$$\Rightarrow \int_V |D_i^h u(x)|^p dx \le \int_\Omega \|Du(y)\|^p dy$$

$$\Rightarrow \sum_{i=1}^n \int_V |D_i^h u(x)|^p dx \le n \int_\Omega \|Du(y)\|^p dy.$$

$$\Rightarrow \|D^h u\|_{L^p(V)} \le C(n,p) \|Du\|_{L^p(\Omega)}$$

For general case, take $u_m \longrightarrow u$ in $W^{1,p}(\Omega)$

$u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$

$$\Rightarrow \|D^h u_m\|_{L^p(V)} \le C(n,p) \|Du_m\|_{L^p(\Omega)}$$

Let $m \to \infty \Rightarrow \|D^h u\|_{L^p(V)} \le C(n,p) \|Du\|_{L^p(\Omega)}$

$$\left(\int_V \left| \frac{u_m(x+he_i) - u_m(x)}{h} \right|^p \longrightarrow \int_V \left| \frac{u(x+he_i) - u(x)}{h} \right|^p \right.$$

b/c $u_m \to u$ in $L^p(\Omega)$

The pf of (ii).

Recall that $1 < p < \infty$. $L^p(\Omega)$ is a reflexive Banach space.

This implies that if $\|u_m\|_{L^p(\Omega)} < \infty$, $m=1,\dots$ (uniform bounded)

$\Rightarrow \exists$ subseq $\{u_{m_j}\}$ $u \in L^p(\Omega)$ s.t. $u_{m_j} \longrightarrow u$ weakly

(i.e. $\displaystyle\int_\Omega u_{m_j} v \xrightarrow[m_j \to \infty]{} \int_\Omega uv$ for $v \in L^q(\Omega)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

$\left(\frac{1}{p}+\frac{1}{q}=1\right)$

Let $\phi \in C_0^\infty(V)$

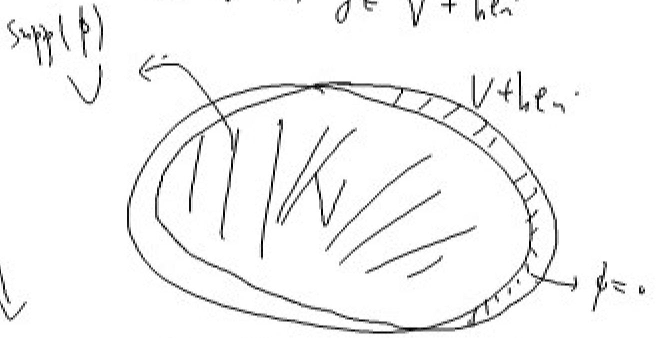
$$\int_V u(x)\left(\frac{\varphi(x+he_i)-\varphi(x)}{h}\right)dx$$

$$=\frac{1}{h}\left(\int_V u(x)\,\phi(\underbrace{x+he_i}_{=y})\,dx-\int_V u(x)\varphi(x)\,dx\right)$$

$$=\frac{1}{h}\int_{V+he_i} u(y-he_i)\,\phi(y)\,dy-\int_V u(y)\phi(y)\,dy$$

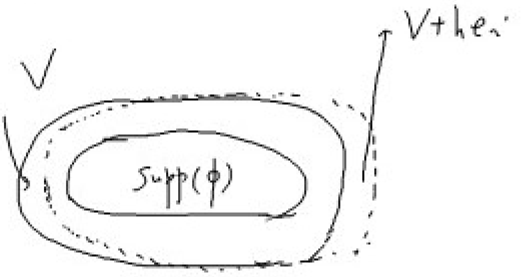
($y=x+he_i$, $x=y-he_i$, h is small enough)

$x\in V \Rightarrow y\in V+he_i$



$$=-\int_V \underbrace{\left[\frac{u(y)-u(y-he_i)}{h}\right]}_{D_i^{-h}u}\phi(y)\,dy$$

$$\Rightarrow \int_V u(D_i^h\phi)\,dx=-\int_V (D_i^{-h}u)\,\phi\,dx$$



If h is small $\Rightarrow$ $\mathrm{supp}(\phi)\subset\subset V+he_i$

$$\int_{V+he_i} u(y-he_i)\,\phi(y)\,dy$$

$$=\int_{\mathrm{supp}(\phi)} u(y-he_i)\,\phi(y)\,dy$$

$$=\int_V u(y-he_i)\,\phi(y)\,dy$$

From assumption, we have $\|D^h u\|_{L^p(V)} \leq C < \infty$

for all $0 < |h| < \frac{1}{2} \operatorname{dist}(V, \partial\Omega)$.

$\Rightarrow \quad \|D^{-h} u\|_{L^p(V)} \leq C$

By remark earlier about $L^p(V)$, we can find a subsequence

$\{D_k^{-h_i} u\}_{i=1}^{\infty}$, $(i \to \infty,\ h_i \to 0)$

$D_k^{-h_i} u \longrightarrow V_k$ as $i \to \infty$ and $V_k \in L^p(V)$

(weak limit of $D_k^{-h_i} u$) $\|V_k\|_{L^p(V)} \leq \underline{\lim} \|D_k^{-h_i} u\|$

But then $\displaystyle\int_V u \frac{\partial \phi}{\partial x_k}\, dx \quad (\phi \in C_0^\infty(V)$ $\subset L^q(V)$, $\frac{1}{p} + \frac{1}{q} = 1)$

$$= \lim_{i \to \infty} \int_V u\, D_k^{h_i} \phi\, dx$$

$$= \lim_{i \to \infty} (-1) \int_V (D_k^{-h_i} u)\, \phi\, dx$$

$$= (-1) \int_V V_k\, \phi\, dx \quad \left(\because D_k^{-h_i} u \to V_k \text{ in } L^p(V)\right)$$

$$\Rightarrow \int_V u \frac{\partial \phi}{\partial x_k}\, dx = (-1) \int_V \underline{\underline{V_k}}\, \phi\, dx$$

So $V_k = D^k u$ (in the weak sense)

and $\|V_k\|_{L^p(V)} = \|D^k u\|_{L^p(V)} \leq \underline{\lim}_{i \to \infty} \|D_k^{-h_i} u\| \leq C$

$\Rightarrow \quad \|Du\|_{L^p(V)} \leq C$ (By assumption, we have $u \in L^p(V)$.)

$\Rightarrow \quad u \in W^{1,p}(V)$