PDE Notes: 09/24/08

Notes will be available at

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⇒ Link to this class

⇒ HW and schedule.

Th: $\sup_{B_r} \frac{|\nabla u|}{u} \le \frac{C}{r}$

for all harmonic ftns $u: B_{2r} \to \mathbb{R}$.

Last time, we showed that

$$\left(\phi : B_2 \to \mathbb{R} \text{ with } \phi\big|_{\partial B_2} = 0, \quad \phi > 0 \text{ in } B_2 \right)$$

$$v = \ln u, \quad w = |\nabla v|^2 \left(= \frac{|\nabla u|^2}{u^2} \right)$$

$$\Delta(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4)$$

$$= 2\phi^4 |\mathrm{Hess}(v)|^2 + \underbrace{2\nabla\phi^4 \cdot \nabla w} + w\Delta\phi^4 + 2w\nabla v \cdot \nabla\phi^4$$

Compute $2\nabla\phi^4 \cdot \nabla w$

$$= \sum_i 2 \frac{\partial}{\partial x_i}(\phi^4) \cdot \frac{\partial}{\partial x_i} \Big(\overbrace{\sum_j \left(\frac{\partial v}{\partial x_j}\right)^2}^{w} \Big)$$

$$= 4 \sum_{i,j} \frac{\partial}{\partial x_i}(\phi^4) \frac{\partial v}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$$

$$\Rightarrow \quad \Delta(w\phi^4) + 2\nabla V \cdot \nabla(w\phi^4)$$

$$= \sum_{i,j} 2\phi^4 \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)^2 + 4 \frac{\partial \phi^4}{\partial x_i} \frac{\partial V}{\partial x_j} \frac{\partial^2 V}{\partial x_i \partial x_j} + w \Delta \phi^4 + 2 w \nabla V \cdot \nabla \phi^4$$

Let $a_{ij} = \phi^2 \frac{\partial^2 V}{\partial x_i \partial x_j}$ and $b_{ij} = \frac{\partial \phi^4}{\partial x_i} \cdot \frac{\partial V}{\partial x_j}$

$$\begin{cases} a_{ij}^2 + 4 a_{ij} b_{ij} \geq -4 b_{ij}^2 \\ \sum_{i,j} b_{ij}^2 = |\nabla \phi^4|^2 |\nabla V|^2 \end{cases}$$

$$\left(\sum_{i,j} b_{ij}^2 = \sum_{i,j} \left(\frac{\partial \phi^4}{\partial x_i}\right)^2 \left(\frac{\partial V}{\partial x_j}\right)^2 = \sum_i \left(\frac{\partial \phi^4}{\partial x_i}\right)^2 \cdot \sum_j \left(\frac{\partial V}{\partial x_j}\right)^2 \right.$$

$$= |\nabla \phi^4|^2 |\nabla V|^2$$

$$\Rightarrow \quad \Delta(w\phi^4) + 2\nabla V \cdot \nabla(w\phi^4) = \sum_{i,j} \phi^4 \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)^2 + a_{ij}^2 + 4 a_{ij} b_{ij} + w \Delta \phi^4 + 2 w \nabla V \cdot \nabla \phi^4$$

$$\Delta(w\phi^4) + 2\nabla v\cdot\nabla(w\phi^4)$$

$$\Rightarrow = \sum_{ij}\phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 + a_{ij}^2 + 4a_{ij}b_{ij} + w\Delta\phi^4 + 2w\nabla v\cdot\nabla\phi^4$$

$$\geq \sum_{ij}\phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 4b_{ij}^2 + w\Delta\phi^4 + 2w\nabla v\cdot\nabla\phi^4$$

$$= \sum_{ij}\phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 4|\nabla\phi^4|^2|\nabla v|^2 + w\Delta\phi^4 + 2w\nabla v\cdot\nabla\phi^4$$

$$\nabla\phi^4 = 4\phi^3\nabla\phi, \quad \nabla v\cdot\nabla\phi^4 = 4\phi^3\nabla v\cdot\nabla\phi$$

$$\Delta\phi^4 = \mathrm{div}(\nabla\phi^4) = \mathrm{div}(4\phi^3\cdot\nabla\phi)$$

$$= 12\phi^2|\nabla\phi|^2 + 4\phi^3\Delta\phi$$

$$|\nabla\phi^4|^2 = 16\phi^6|\nabla\phi|^2$$

$$= \sum_{ij}\phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 64\phi^6|\nabla\phi|^2|\nabla v|^2$$

$$+ 12w\phi^2|\nabla\phi|^2 + 4w\phi^3\Delta\phi + \underbrace{8w\phi^3\nabla v\cdot\nabla\phi}_{\geq}$$

$$\geq \sum_{ij}\phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 64\phi^4|\nabla\phi|^2|\nabla v|^2$$

$$+ 12w\phi^2|\nabla\phi|^2 + 4w\phi^3\Delta\phi - \underline{8w\phi^3|\nabla v||\nabla\phi|}$$

$$= \sum_{ij}\phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - 64\phi^6|\nabla\phi|^2|\nabla v|^2$$

$$+ \left(12\phi^2|\nabla\phi|^2 + 4\phi^3\Delta\phi\right)w - 8\phi^3 w|\nabla v||\nabla\phi|$$

b/c ϕ is a $C^2(\overline{B_{2r}})$, $\exists\ c_1>0, c_2>0, c_3>0$

$$\Rightarrow -64\phi^4|\nabla\phi|^2 \geq -c_1$$

$$\Rightarrow 4\phi\Delta\phi + 12|\nabla\phi|^2 \geq -c_2$$

$$-8|\nabla\phi| \geq -c_3$$

$$\geq \sum_{ij}\phi^4\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2 - c_1\phi^2|\nabla v|^2 - c_2\phi^2 w - c_3\phi^3 w|\nabla v|$$

$$\frac{\sum_{ij}\left(\frac{\partial^2 v}{\partial x_i\partial x_j}\right)^2}{n} \geq \frac{\sum_{i=1}^{n}\left(\frac{\partial^2 v}{\partial x_i^2}\right)^2}{n} \geq \left(\frac{\sum_i \frac{\partial^2 v}{\partial x_i^2}}{n}\right)^2$$

$$= \left(\frac{\Delta v}{n}\right)^2 n$$

$$\geq \phi^4\frac{(\Delta v)^2}{n} - c_1\phi^2|\nabla v|^2 - c_2\phi^2 w - c_3\phi^3 w|\nabla v|$$

(Recall $\Delta v = -w$, $|\nabla v|^2 = w$

$$\Rightarrow \Delta(w\phi^4) + 2\nabla V \cdot \nabla(w\phi^4)$$

$$* \left(\geq \frac{\phi^4}{n} w^2 - c_1\phi^2 w - c_2\phi^2 w - c_3\phi^3 w^{\frac{3}{2}} \right.$$

b/c $w > 0$ and $\phi > 0$ in B_{2r} ($\phi = 0$ on ∂B_{2r}).

$$\left| \begin{array}{l} w\phi^4 > 0 \text{ in } B_{2r} \\ w\phi^4 = 0 \text{ on } \partial B_{2r} \end{array} \right.$$

b/c $w\phi^4 \in C^2(\overline{B_2}) \Rightarrow w\phi^4$ must achieve an interior maximum at x_0

$$\Rightarrow \left| \begin{array}{l} \Delta(w\phi^4)(x_0) \leq 0 \\ \nabla(w\phi^4)(x_0) = 0 \end{array} \right.$$