Gradient Estimate

PDE NOTES
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Th: There are dimension . . . . . , $C = C(n)$
such that
$$\sup_{B_r} \frac{|\nabla u|}{u} \le \frac{C}{r}$$
for all postive harmonic ftns $u : B_{2r} \to \mathbb{R}$

Maximum Principle for C^2 ftns:

$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ where Ω is bounded.

Suppose $u(x_0) = \max_{\Omega} u$ where $x_0 \in \Omega$

(ie u achieves an interior maximum)

$$\Rightarrow \begin{cases} \nabla u(x_0) = 0 \\ \Delta u(x_0) \le 0 \end{cases}$$

$(u_{ij}) \le 0$ at x_0

(It follows from 2nd derivative test

$\Rightarrow \mathrm{Hess}(u)(x_0) = (u_{ij}(x_0)) \le 0$ nonpostive definite

and $\Delta u(x_0) = \mathrm{trace}((u_{ij}(x_0))) \le 0$

(Bochner formula)

Prop: Let $u \in C^2(\Omega)$.

Then $\frac{1}{2} \Delta |\nabla u|^2 = \langle \nabla \Delta u, \nabla u \rangle + |\mathrm{Hess}\, u|^2$

where $\langle \; , \; \rangle$ is the inner product in $\mathbb{R}^n$

$$|\mathrm{Hess}\, u|^2 = \sum_{i,j=1}^{n} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 \quad \text{and}$$

$$\mathrm{Hess}(u) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n}$$

pf:
$$|\nabla u|^2 = \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j} \right)^2$$

$$\Delta |\nabla u|^2 = \sum_i \frac{\partial^2}{\partial x_i^2} \left(\sum_j \left(\frac{\partial u}{\partial x_j} \right)^2 \right)$$

$$= \sum_i \frac{\partial}{\partial x_i} \cdot \frac{\partial}{\partial x_i} \left(\sum_j \left(\frac{\partial u}{\partial x_j} \right)^2 \right)$$

$$= \sum_{i,j} \frac{\partial}{\partial x_i} \left(2 \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial u}{\partial x_j} \right)$$

$$= \sum_{i,j} \left(2 \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \frac{\partial u}{\partial x_j} + 2 \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right)$$

$$= \sum_{i,j} 2 \frac{\partial}{\partial x_j} \left(\frac{\partial^2 u}{\partial x_i^2} \right) \cdot \frac{\partial u}{\partial x_j} + 2 |\mathrm{Hess}(u)|^2$$

$$= 2 \sum_j \frac{\partial}{\partial x_j} (\Delta u) \cdot \frac{\partial u}{\partial x_j} + 2 |\mathrm{Hess}(u)|^2$$

$$= 2 \langle \nabla \Delta u, \nabla u \rangle + 2 |\mathrm{Hess}(u)|^2$$

$$\Rightarrow \frac{1}{2} \Delta |\nabla u|^2 = \langle \nabla \Delta u, \nabla u \rangle + |\mathrm{Hess}(u)|^2$$

Remark: On a Riemannian manifold

We can define ∇u and Δu

$$\Rightarrow \quad \frac{1}{2} \Delta |\nabla u|^2 = \langle \nabla \Delta u, \nabla u \rangle + |\mathrm{Hess}(u)|^2 + \underline{\mathrm{Ric}}(\nabla u, \nabla u)$$

Ricci Curvature.

Pf of the gradient Estimate:

We prove the result for $r=1$ 1st

Choose $\phi : B_{2r} \to \mathbb{R}$ with $\phi = 0$ on ∂B_{2r}.

Define $\begin{cases} v = \log(u) \\ w = |\nabla v|^2 \left(= \dfrac{|\nabla u|^2}{u^2} \right) \end{cases}$ (b/c $u > 0 \Rightarrow \ln(u)$ is well-defined)

We want to estimate w.

Note that $\nabla v = \nabla(\ln(u)) = \dfrac{\nabla u}{u}$

$$\Delta v = \operatorname{div}(\nabla v) = \operatorname{div}\left(\frac{\nabla u}{u}\right) = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}$$

$$= -\frac{|\nabla u|^2}{u^2} \quad (\text{b/c } \Delta u = 0)$$

$$= -w$$

Compute
$$\Delta w = \Delta\left(|\nabla v|^2\right)$$
$$= 2\langle \nabla \Delta v, \nabla v\rangle + 2|\operatorname{Hess} v|^2$$
$$= -2\langle \nabla w, \nabla v\rangle + 2|\operatorname{Hess}(v)|^2$$
$$(\Delta v = -w)$$

Compute $\Delta(w\phi^4)$
$$= \underline{(\Delta w)\phi^4} + 2\langle \nabla w, \nabla \phi^4\rangle + w\Delta\phi^4$$
$$\left(\Delta(fg) = \Delta f\, g + 2\langle \nabla f, \nabla g\rangle + f\Delta g\right)$$
$$= 2\phi^4|\operatorname{Hess}(v)|^2 - 2\phi^4\langle \nabla w, \nabla v\rangle + 2\langle\nabla w, \nabla\phi^4\rangle + w\Delta\phi^4$$

Note that
$$\langle \nabla(\phi^4 w), \nabla v\rangle$$
$$= w\langle \nabla\phi^4, \nabla v\rangle + \phi^4\langle\nabla w, \nabla v\rangle$$
$$\Rightarrow \quad -\phi^4\langle\nabla w, \nabla v\rangle = -\langle \nabla(\phi^4 w), \nabla v\rangle + w\langle\nabla\phi^4, \nabla v\rangle$$

$$\Rightarrow \quad \Delta(w\phi^4) = 2\phi^4|\operatorname{Hess}(v)|^2 - 2\langle\nabla(\phi^4 w), \nabla v\rangle + 2w\langle\nabla\phi^4, \nabla v\rangle + 2\langle\nabla w, \nabla\phi^4\rangle + w\Delta\phi^4$$

$$\Rightarrow \quad \Delta(w\phi^4) + 2\langle\nabla(\phi^4 w), \nabla v\rangle$$
$$= 2\phi^4|\operatorname{Hess}(v)|^2 + 2\langle\nabla\phi^4, \nabla w\rangle + w\Delta\phi^4 + 2w\langle\nabla v, \nabla\phi^4\rangle$$