Last time, we proved the Poincare's inequality

Th Let $\Omega \subset\subset \mathbb{R}^n$ (open, connected) with C^1 boundary.

Assume $1 \le p < \infty$

Then $\exists$ a constant $C(n,p,\Omega)$ s.t

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \le C \|Du\|_{L^p(\Omega)}$$

for each $u \in W^{1,p}(\Omega)$.

Here $(u)_\Omega = \dfrac{\int_\Omega u(y)dy}{\mathrm{Vol}(\Omega)}$.

We'll use this to prove the following.

Th2 (Poincare's inequality for a ball)

Assume $1 \le p < \infty$

Then there exists a constant $C(n,p)$ s.t

$$\|u - (u)_{B(x,r)}\|_{L^p(B(x,r))} \le \underline{C(n,p)\, r}\, \|Du\|_{L^p(B(x,r))}$$

for $u \in W^{1,p}(B(x,r))$

Remark: From previous Th, we have

$$\|u - (u)_{B(x,r)}\|_{L^p(B(x,r))} \le \underset{\substack{\| \\ C(n,p)\, r \leftarrow \text{From this Th.}}}{\tilde{C}(n,p,B(x,r))} \|Du\|_{L^p(B(x,r))}$$

pf: $u \in W^{1,p}(B(0,1))$

By previous Th,

$$\|u - (u)_{B(0,1)}\|_{L^p(B(0,1))} \le \underset{\substack{\| \\ \tilde{C}(n,p)}}{C(n,p,B(0,1))} \|Du\|_{L^p(B(0,1))}$$

Suppose $u \in W^{1,p}(B(x,r))$.

Let $v(y) = u(x+ry)$.

Then $v \in W^{1,p}(B(0,1))$

$$\Rightarrow \|v - (v)_{B(0,1)}\|_{L^p(B(0,1))} \le \tilde{C}(n,p) \|Dv\|_{L^p(B(0,1))}$$

$$\|Dv\|_{L^p(B(0,1))} = \sum_{|\alpha|=1} \|D^\alpha v\|_{L^p(B(0,1))}$$

$$= \sum_{|\alpha|=1} \left(\int_{B(0,1)} |D^\alpha v(y)|^p\, dy \right)^{\frac{1}{p}}$$

$$\left(\begin{array}{l} D^\alpha v(y) = D^\alpha_y (u(x+ry)) \\ \quad |\alpha|=1 \\ \quad = r (D^\alpha_x u)(x+ry) \\ \text{Let } z = x+ry \\ \Rightarrow z \in B(x,r),\ dz = r^n dy \end{array} \right) \quad \left(\begin{array}{l} \frac{\partial}{\partial y_j} u(\underline{\underline{x+ry}}) \\ = \frac{\partial u}{\partial z_j}(x+ry) \cdot r \end{array} \right)$$

$$= \sum_{|\alpha|=1} \left(\int_{B(x,r)} (r |D^\alpha u(z)|)^p\, r^{-n} dz \right)^{\frac{1}{p}}$$

$$= \sum_{|\alpha|=1} r\, r^{-\frac{n}{p}} \left(\int_{B(x,r)} |D^\alpha u(z)|^p\, dz \right)^{\frac{1}{p}}$$

$$= r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

$$(v)_{B(0,1)} = \frac{\int_{B(0,1)} u(x+ry)\, dy}{\mathrm{Vol}(B(0,1))} = \frac{\int_{B(x,r)} u(z) r^{-n} dz}{\mathrm{Vol}(B(0,1))}$$

$$= \frac{\int_{B(x,r)} u(z)\, dz}{r^n\, \mathrm{Vol}(B(0,1))} = (u)_{B(x,r)}$$

$$\|v-(v)_{B(0,1)}\|_{L^p(B(0,1))}$$

$$= \left(\int_{B(0,1)} |u(x+ry) - (u)_{B(x,r)}|^p \, dy\right)^{\frac{1}{p}}$$

$$z = x+ry, \quad dz = r^n dy$$

$$= \left(\int_{B(x,r)} |u(z) - (u)_{B(x,r)}|^p \, r^{-n} dz\right)^{\frac{1}{p}}$$

$$= r^{-\frac{n}{p}} \|u-(u)_{B(x,r)}\|_{L^p(B(x,r))}.$$

So we have $\|Dv\|_{L^p(B(0,1))} = r^{1-\frac{n}{p}}\|Du\|_{L^p(B(x,r))}$

and $\|v-(v)_{B(0,1)}\|_{L^p(B(0,1))} = r^{-\frac{n}{p}}\|u-(u)_{B(x,r)}\|_{L^p(B(x,r))}$

Then $\|v-(v)_{B(0,1)}\|_{L^p(B(0,1))} \le \tilde{C}(n,p)\|Dv\|_{L^p(B(0,1))}$

$$\Rightarrow \quad r^{-\frac{n}{p}}\|u-(u)_{B(x,r)}\|_{L^p(B(x,r))} \le \tilde{C}(n,p)\, r^{1-\frac{n}{p}}\|Du\|_{L^p(B(x,r))}$$

$$\Rightarrow \quad \|u-(u)_{B(x,r)}\|_{L^p(B(x,r))} \le \tilde{C}(n,p)\, r\, \|Du\|_{L^p(B(x,r))}$$

$BMO(\mathbb{R}^n)$ = The space of bounded mean oscillation

$$= \left\{u \;\middle|\; u \in L^1_{loc}(\mathbb{R}^n),\ \sup_{B(x,r)\subset\mathbb{R}^n} \fint_{B(x,r)} |u-(u)_{B(x,r)}|\,dy < \infty\right\}$$

Claim: $W^{1,n}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$

pf: $u \in W^{1,n}(\mathbb{R}^n)$

$\Rightarrow \quad u \in W^{1,n}(\underline{B(x,r)}) \subset W^{1,1}(B(x,r))$

By previous Th, $\|u-(u)_{B(x,r)}\|_{L^1(B(x,r))} \le Cr\|Du\|_{L^1(B(x,r))}$
($p=1$)

$$\le Cr\left(\int_{B(x,r)} |Du(y)|\,dy\right)$$

$$\le Cr\left(\int_{B(x,r)} |Du(y)|^n\,dy\right)^{\frac{1}{n}} \cdot \underbrace{\left(\int_{B(x,r)} 1\right)^{\frac{n-1}{n}}}_{(\alpha_n)^{\frac{n-1}{n}}\, r^{n-1}}$$

$$\le \tilde{C}\, r^n \|Du\|_{L^n(B(x,r))}$$

$$\Rightarrow \quad \frac{\|u-(u)_{B(x,r)}\|_{L^1(B(x,r))}}{Vol(B(x,r))} \le \hat{\tilde{C}}(n,p)\|Du\|_{L^n(B(x,r))}$$

$$\fint_{B(x,r)} |u-(u)_{B(x,r)}| \le \hat{C}(n,p)\|Du\|_{L^n(\mathbb{R}^n)}$$

$$\le \hat{C}(n,p)\,\underline{\underline{\|u\|_{W^{1,n}(\mathbb{R}^n)}}}$$

$$< \infty$$

$u \in W^{1,n}(\mathbb{R}^n)$

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