§ 5.8.2 Difference quotients

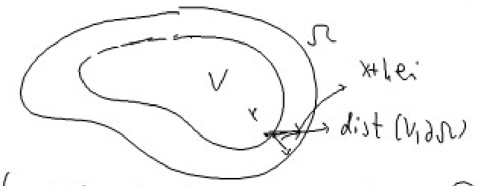
We'll use difference quotient to approximate weak derivatives later on.

Here we study the difference quotients $u:\Omega \longrightarrow \mathbb{R}$

Def: (i) The i-th difference quotient of size h is

$$D_i^h u(x) := \frac{u(x+he_i)-u(x)}{h} \quad (i=1,\cdots,n)$$

for $x \in V$ and $0<|h|<\operatorname{dist}(V,\partial\Omega)$



$(x \in V,\ |h|<\operatorname{dist}(V,\partial\Omega) \Rightarrow x+he_i \in \Omega)$

(ii) $D^h u := (D_1^h u, D_2^h u, \cdots, D_n^h u)$.

Th 3 (Difference quotients and weak derivatives)

(i) Suppose $1 \le p < \infty$ and $u \in W^{1,p}(\Omega)$.
Then for each $V \subset\subset \Omega$

$$\Rightarrow \quad \|D^h u\|_{L^p(V)} \le C \|Du\|_{L^p(\Omega)}$$

for some constant $C(n,p)$ and
$0<|h|<\frac{1}{2}\operatorname{dist}(V,\partial\Omega)$

(ii) Assume $1<p<\infty$, $u \in L^p(V)$ and there exists a constant C such that

$$\|D^h u\|_{L^p(V)} \le C \quad (\text{indep of } h)$$

for all $0<|h|<\frac{1}{2}\operatorname{dist}(V,\partial\Omega)$.

Then $u \in W^{1,p}(V)$ with $\|Du\|_{L^p(V)} \le C$

pf: 1° Assume u is smooth first.

$$u(x+he_i)-u(x) = \int_0^1 \left(\frac{d}{dt} u(x+the_i)\right) dt$$

$$= \int_0^1 (\nabla u(x+the_i)\cdot he_i)\, dt, \quad \|e_i\|=1$$

$$\Rightarrow |u(x+he_i)-u(x)| \le |h| \int_0^1 \|Du(x+the_i)\|\, dt$$

$$\left|\frac{u(x+he_i)-u(x)}{h}\right| \le \int_0^1 \|Du(x+the_i)\|\, dt$$

$$\Big\Downarrow \quad \left(0<|h|<\tfrac{1}{2}\operatorname{dist}(V,\partial\Omega)\right)$$

$$|D_i^h u(x)| \le \int_0^1 \|Du(x+the_i)\|\, dt$$

$$\le \left(\int_0^1 \|Du(x+the_i)\|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 1^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}}$$

$$\le \left(\int_0^1 \|Du(x+the_i)\|^p dt\right)^{\frac{1}{p}}$$

$$\Rightarrow \quad |D_i^h u(x)|^p \le \int_0^1 \|Du(x+the_i)\|^p dt$$

$$\Rightarrow \quad \int_V |D_i^h u(x)|^p dx \le \int_V \int_0^1 \|Du(x+the_i)\|^p dt\, dx$$

$$= \int_0^1 \int_V \|Du(x+the_i)\|^p dx\, dt$$

$$\le \int_0^1 \int_\Omega \|Du(y)\|^p dy\, dt \quad \left(\overset{y}{\overset{\parallel}{x+the_i}} \in \Omega\right)$$

$$\Rightarrow \int_V |D_i^h u(x)|^p dx \le \int_\Omega \|Du(y)\|^p dy$$

$$\Rightarrow \sum_{i=1}^n \int_V |D_i^h u(x)|^p dx \le n \int_\Omega \|Du(y)\|^p dy.$$

$$\Rightarrow \|D^h u\|_{L^p(V)} \le C(n,p) \|Du\|_{L^p(\Omega)}$$

For general case, take $u_m \longrightarrow u$ in $W^{1,p}(\Omega)$, $u_m \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$

$$\Rightarrow \|D^h u_m\|_{L^p(V)} \le C(n,p) \|Du_m\|_{L^p(\Omega)}$$

Let $m \to \infty \Rightarrow \|D^h u\|_{L^p(V)} \le C(n,p) \|Du\|_{L^p(\Omega)}$

$$\left(\int_V \left|\frac{u_m(x+he_i) - u_m(x)}{h}\right|^p \longrightarrow \int_V \left|\frac{u(x+he_i) - u(x)}{h}\right|^p \right.$$

$\because u_m \to u$ in $L^p(\Omega)$

The pf of (ii).

Recall that $1 < p < \infty$. $L^p(\Omega)$ is a reflexive Banach space.

This implies that if $\|u_m\|_{L^p(\Omega)} < \infty$, $m=1,\cdots$ (uniform bounded)

$\Rightarrow \exists$ subseq $\{u_{m_j}\}$, $u \in L^p(\Omega)$ st $u_{m_j} \longrightarrow u$ weakly

(i.e. $\displaystyle\int_\Omega u_{m_j} v \xrightarrow[m_j \to \infty]{} \int_\Omega uv$ for $v \in L^q(\Omega)$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

Let $\phi \in C_c^\infty(V)$

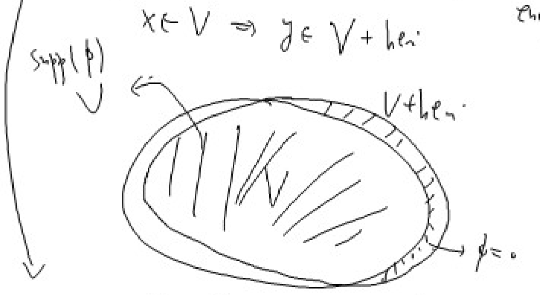
$$\int_V u(x) \left(\frac{\varphi(x+he_i) - \varphi(x)}{h} \right) dx$$

$$= \frac{1}{h} \left(\int_V u(x) \phi(\underbrace{x+he_i}_{=y}) dx - \int_V u(x) \varphi(x) dx \right)$$

$$= \frac{1}{h} \int_{V+he_i} u(y-he_i) \phi(y) dy - \int_V u(y) \phi(y) dy$$

($y = x + he_i$, $x = y - he_i$, h is small enough)

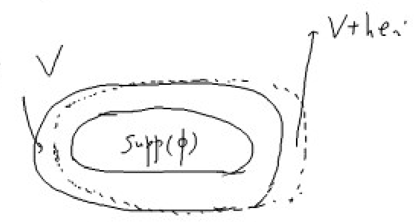
$x \in V \Rightarrow y \in V + he_i$



$$= -\int_V \frac{[u(y) - u(y-he_i)]}{h} \phi(y) dy$$

$D_i^{-h} u$

$$\Rightarrow \int_V u(D_i^h \phi) dx = -\int_V (D_i^{-h} u) \phi dx$$



If h is small $\Rightarrow$ supp$(\phi) \subset\subset V + he_i$

$$\int_{V+he_i} u(y-he_i) \phi(y) dy$$

$$= \int_{\text{supp}(\phi)} u(y-he_i) \phi(y) dy$$

$$= \int_V u(y-he_i) \phi(y) dy$$

From assumption, we

have $\|D^h u\|_{L^p(V)} \le C < \infty$

for all $0 < |h| < \frac{1}{2} \operatorname{dist}(V, \partial\Omega)$.

$\Rightarrow \quad \|D^{-h} u\|_{L^p(V)} \le C$

By remark earlier about $L^p(V)$,

we can find a subsequence

$\{ D_k^{-h_i} u \}_{i=1}^{\infty}$, $(i \to \infty,\ h_i \to 0)$

$D_k^{-h_i} u \longrightarrow V_k$ as $i \to \infty$ and $V_k \in L^p(V)$

(weak limit of $D_k^{-h_i} u$) $\|V_k\|_{L^p(V)} \le \underline{\lim} \|D_k^{-h_i} u\|$

But then $\int_V u \frac{\partial \phi}{\partial x_k} dx \quad (\phi \in C_0^\infty(V))$

$$= \lim_{i \to \infty} \int_V u\, D_k^{h_i} \phi\, dx \qquad \begin{array}{c} \cap \\ L^q(V) \\ \frac{1}{p} + \frac{1}{q} = 1 \end{array}$$

$$= \lim_{i \to \infty} (-1) \int_V (D_k^{-h_i} u)\, \phi\, dx$$

$$= (-1) \int_V V_k\, \phi\, dx \quad \left(b/c \quad D_k^{-h_i} u \rightharpoonup \underset{L^p(V)}{V_k} \right)$$

$$\Rightarrow \int_V u \frac{\partial \phi}{\partial x_k} dx = (-1) \int_V \underline{\underline{V_k}}\, \phi\, dx$$

So $V_k = D^k u$ (in the weak sense)

and $\|V_k\|_{L^p(V)} = \|D^k u\|_{L^p(V)} \le \underline{\lim}_{i \to \infty} \|D_k^{-h_i} u\| \le C$

$\Rightarrow \quad \|Du\|_{L^p(V)} \le C$ (By assumption, we have $u \in L^p(V)$.)

$\Rightarrow \quad u \in W^{1,p}(V)$