① Recall that $\Omega \subset\subset \mathbb{R}^n$, $1 \le p < n$

$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$

for $1 \le q < p^* = \frac{np}{n-p}$

b/c $p^* - p = \frac{np}{n-p} - p = \frac{np - np + p^2}{n-p} = \frac{p^2}{n-p} > 0$

$\Rightarrow 1 \le p < p^*$

$\Rightarrow W^{1,p}(\Omega) \subset\subset L^p(\Omega)$

when $1 \le p < n$.

② When $p = n$

b/c Ω is bounded

$W^{1,n}(\Omega) \subset W^{1,r}(\Omega)$ for $1 \le r < n$

by Hölder inequality

(b/c $L^s(\Omega) \subset L^t(\Omega)$ if $1 \le t \le s$)

Ω is bounded.

$r^* = \frac{rp}{p-r} \Rightarrow r^* \to \infty$ as $r \to p^-$

$W^{1,n}(\Omega) \subset W^{1,r}(\Omega) \subset\subset L^q(\Omega)$

$1 \le q < r^*$

We can find $r < p$ and close to p

s.t. $n < r^*$. Choose $q = n$

$\Rightarrow W^{1,n}(\Omega) \subset W^{1,r}(\Omega) \subset\subset L^n(\Omega)$

for this choice of r.

$\Rightarrow W^{1,n}(\Omega) \subset\subset L^n(\Omega)$

when $p = n$.

③ $n < p \le \infty$

By Morrey's inequality, we have

$\|u\|_{C^{0,\gamma}(\bar\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}$

$\|$

$\|u\|_{C^0(\bar\Omega)} + \sup\limits_{\substack{x \ne y \\ x,y \in \Omega}} \frac{|u(x)-u(y)|}{\|x-y\|^\gamma}$ where $\gamma = 1 - \frac{n}{p}$

$(0 < \gamma \le 1)$

Claim: $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ when $n < p \le \infty$

From Sobolev inequality, we know

$\underline{W^{1,p}(\Omega) \subset C^{0,\gamma}}(\bar\Omega) \subset L^p(\Omega)$

To show $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$,

it suffices to prove that

if $\{u_n\}_{n=1}^\infty$ is bounded in $W^{1,p}(\Omega)$

$\Rightarrow \exists$ subsequence $\{u_{n_j}\}_{n_j=1}^\infty$ s.t. it

converges in $L^p(\Omega)$

By Sobolev ineq,

$\Rightarrow \underline{\|u_n\|_{C^0(\bar\Omega)} + \sup\limits_{\substack{x,y \in \Omega \\ x \ne y}} \frac{|u_n(x)-u_n(y)|}{\|x-y\|^\gamma}} < C$

$\Rightarrow \{u_n\}$ is uniformly bounded

$0 < \gamma < 1$

$|u_n(x) - u_n(y)| < C\|x-y\|^\gamma$ $(x \ne y)$

$\Rightarrow \{u_n\}$ is equi-continuous

By Arzela-Ascoli Th

$\{u_m\}$ has a convergent subsequence

$\{u_{m_j}\}_{m_j=1}^{\infty}$ in $C^0(\bar{\Omega})$

So this subsequence also converges in $L^p(\Omega)$.

#.

Remark: A bounded set in $C^{k,r}(\bar{\Omega})$ is precompact.

So we have $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$ ~~for all~~ p.

$\left(\Rightarrow \|u\|_{L^p(\Omega)} \le C \|u\|_{W^{1,p}(\Omega)}\right)$

Notation: $(u)_\Omega = \dfrac{\int_\Omega u}{Vol(\Omega)}$ = average value of u on Ω

Th (Poincare's inequality)

Let Ω be a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary. Ω is connected

Assume $1 \le p \le \infty$

Then $\exists$ $C = C(n, p, \Omega)$ s.t

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \le C \|Du\|_{L^p(\Omega)}$$

for each $u \in W^{1,p}(\Omega)$

Remark. This bound only depends on $\|Du\|_{L^p(\Omega)}$.

Pf: We'll prove this by contradiction

If the statement (inequality) were false,

$\Rightarrow \exists$ $\{u_k\}_{k=1}^{\infty}$ s.t

$$\begin{cases} u_k \in W^{1,p}(\Omega) \text{ and} \\ \|u_k - (u_k)_\Omega\|_{L^p(\Omega)} > k \|Du_k\|_{L^p(\Omega)} \end{cases}$$

We normalize by defining

$$v_k := \frac{u_k - (u_k)_\Omega}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}}$$

Then $\|v_k\|_{L^p(\Omega)} = 1$, $(v_k)_\Omega = 0$

$$Dv_k = \frac{Du_k}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}}$$

$$\|Dv_k\|_{L^p(\Omega)} = \frac{\|Du_k\|_{L^p(\Omega)}}{\|u_k - (u_k)_\Omega\|_{L^p(\Omega)}} < \frac{1}{k}$$

$\Longrightarrow$ $\|v_k\|_{W^{1,p}(\Omega)} = \|v_k\|_{L^p(\Omega)} + \sum_{|\alpha|=1} \|D^\alpha v_k\|_{L^p(\Omega)}$

$\Rightarrow$ $\{v_k\}_{k=1}^{\infty}$ is a bounded seq in $W^{1,p}(\Omega)$

Recall that $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$

$\Rightarrow \exists$ subsequence $\{v_{k_j}\}_{k_j=1}^{\infty}$ st $v_{k_j} \to v$ in $L^p(\Omega)$.

Since $\|v_{k_j}\|_{L^p} = 1$ and $(v_{k_j})_\Omega = 0$

$\Rightarrow$ $\|v\|_{L^p} = 1$ and $(v)_\Omega = 0$

Recall that $\|DV_{k_j}\| < \frac{1}{k_j} \to 0$ as $k_j \to \infty$

and $\|V_{k_j} - V\|_{L^1(\Omega)} \to 0$

(b/c $1 \le p$ and Ω is bounded)

$\phi \in C_0^\infty(\Omega)$ $\quad \|V_{k_j} - V\|_{L^p(\Omega)} \to 0$

$$\Rightarrow \int_\Omega V \frac{\partial \phi}{\partial x_n}\, dx = \lim_{k_j \to \infty} \int_\Omega V_{k_j} \left(\frac{\partial \phi}{\partial x_n}\right)$$

$$\underset{\text{b/c } V_{k_j} \in W^{1,p}(\Omega)}{=} \lim_{k_j \to \infty} (-1) \int_\Omega \phi \underbrace{\frac{\partial}{\partial x_n} V_{k_j}}_{\text{weak derivative}}$$

$$= 0 \quad \left(\text{b/c } \left\|\frac{\partial V_{k_j}}{\partial x_n}\right\|_{L^p(\Omega)} \to 0 \Rightarrow \left\|\frac{\partial V_{k_j}}{\partial x_n}\right\|_{L^1(\Omega)} \to 0 \right)$$

$$\Rightarrow \underbrace{\frac{\partial V}{\partial x_n}}_{\text{weak derivative}} = 0$$

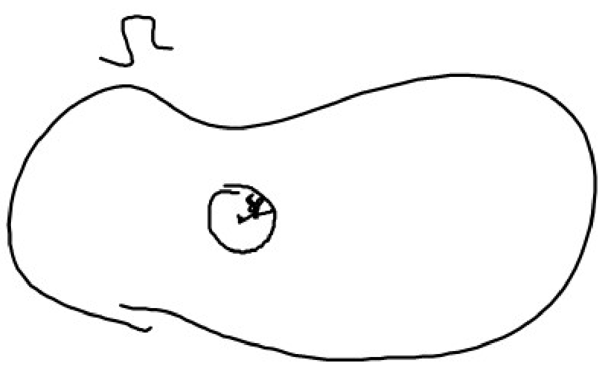
$\Rightarrow DV = 0$. b/c Ω is connected

$\Rightarrow V$ is constant a.e.

Since $(V)_\Omega = 0 \Rightarrow V \equiv 0$ a.e.

This contradicts with the fact that $\|V\|_{L^p(\Omega)} = 1$.

Remark: $Du = 0$ $\quad u \in W^{1,p}(\Omega)$



$$\underbrace{D(u^\varepsilon)}_{\text{smooth}} = (Du)^\varepsilon \quad \text{on } B(x,r) \subset\subset \Omega$$
$$= 0 \text{ if } Du = 0$$

$\Rightarrow u^\varepsilon = C(\varepsilon)$ constant

$u^\varepsilon \xrightarrow[\text{a.e.}]{} u$ as $\varepsilon \to 0$

$\Rightarrow u \equiv$ const a.e. in $B(x, r)$

b/c Ω is connected $\Rightarrow u \equiv$ Const a.e.