Continuation from Last time,

Recall that we have proved

Cor1 : $u \in W^{1,p}(V)$, $\mathrm{supp}(u) \subset\subset V \subset\subset \mathbb{R}^n$

$\Rightarrow \|u - u^\varepsilon\|_{L^1(V)} \leq \varepsilon\, C(V,n,p) \|Du\|_{L^p(V)}$

for ε small enough.

(So $\mathrm{supp}(u^\varepsilon) \subset\subset V$)

Now, we'll prove the following Lemma:

Lemma 2 : Suppose $\{u_m\}_{m=1}^\infty$ where $\mathrm{supp}(u_m) \subset\subset V \subset\subset \mathbb{R}^n$ and

$\exists\, C$ s.t. $\|u_m\|_{L^1(V)} < C$ for all m.

$\Rightarrow$ For each $\varepsilon > 0$ small enough

the seq $\{u_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded

and equi-continuous

Remark: Under the same condition,

b/c $\{u_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded and

equi-continuous. Also $u_m^\varepsilon \in C_0^\infty(V)$ for

ε small enough.

$\Rightarrow$ By Arzela-Ascoli Th

For each ε small enough $\{u_m^\varepsilon\}_{m=1}^\infty$ has

a convergent subsequence $\{u_{m_j}^\varepsilon\}_{j=1}^\infty$ in $C^0(\overline{V})$

$\Rightarrow$ b/c $\overline{V}$ is compact

So $\{u_{m_j}^\varepsilon\}_{j=1}^\infty$ also converge in $L^q(V)$

for $1 \leq q \leq \infty$.

pf: Recall $u_m^\varepsilon(x) = \displaystyle\int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, u_m(y)\, dy$

$$= \int_{B(x,\varepsilon)} \left(\frac{\eta\left(\frac{x-y}{\varepsilon}\right)}{\varepsilon^n} \right) u_m(y)\, dy$$

$$\Rightarrow \quad |u_m^\varepsilon(x)| \leq \frac{\|\eta\|_\infty}{\varepsilon^n} \int_{B(x,\varepsilon)} |u_m(y)|\, dy$$

$$\leq \frac{\|\eta\|_\infty}{\varepsilon^n} \|u_m\|_{L^1(V)}$$

$$\leq \frac{C\|\eta\|_\infty}{\varepsilon^n} \quad (\text{b/c } \|u_m\|_{L^1(V)} < C)$$

$$\Rightarrow \quad \|u_m^\varepsilon\|_\infty \leq \frac{C\|\eta\|_\infty}{\varepsilon^n}$$

$\Rightarrow \{u_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded for each $\varepsilon > 0$ (small enough)

To show that $\{u_m^\varepsilon\}_{m=1}^\infty$ is equi-continuous,

it suffices to prove that $\{Du_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded.

$$Du_m^\varepsilon(x) = \int_{B(x,\varepsilon)} D\eta_\varepsilon(x-y)\, u_m(y)\, dy \qquad \left(\eta_\varepsilon(x) = \frac{\eta\left(\frac{x}{\varepsilon}\right)}{\varepsilon^n},\quad D\eta_\varepsilon = \frac{(D\eta)\left(\frac{x}{\varepsilon}\right)}{\varepsilon^{n+1}} \right)$$

$$= \frac{1}{\varepsilon^{n+1}} \int_{B(x,\varepsilon)} (D\eta)\left(\frac{x-y}{\varepsilon}\right) u_m(y)\, dy$$

$$\|Du_m^\varepsilon\|_\infty \leq \frac{\|D\eta\|_\infty}{\varepsilon^{n+1}} \int_{B(x,\varepsilon)} |u_m(y)|\, dy$$

$$\leq \frac{\|D\eta\|_\infty}{\varepsilon^{n+1}} \|u_m\|_{L^1(V)} \leq \frac{C\|D\eta\|_\infty}{\varepsilon^{n+1}}$$

$\Rightarrow \{Du_m^\varepsilon\}_{m=1}^\infty$ is uniformly bounded for $\varepsilon > 0$ and small enough.

(pf of the compactness Th) $(1 \le p < n)$

$\{u_m\}_{m=1}^{\infty}$ is a bounded sequence in $W^{1,p}(\Omega)$

Slect $V \subset\subset \mathbb{R}^n$ st $\Omega \subset\subset \underset{\text{bounded.}}{\underline{V}} \subset\subset \mathbb{R}^n$.

Let $\{\bar{u}_m\}_{m=1}^{\infty}$ be the extension of u_m st

(i) $\text{supp}(\bar{u}_m) \subset\subset V$ (iii) $\bar{u}_m = u_m$ in Ω

(ii) $\bar{u}_m \in W^{1,p}(\mathbb{R}^n)$ $(\bar{u}_m \in W^{1,p}(V))$

$$\|\bar{u}_m\|_{W^{1,p}(V)} = \|\bar{u}_m\|_{W^{1,p}(\mathbb{R}^n)} \le C(p,\Omega)\, \|u_m\|_{W^{1,p}(\Omega)}$$

$\because$ $\{u_m\}_{m=1}^{\infty}$ is a bounded seq in $W^{1,p}(\Omega)$

$\Rightarrow$ $\{\bar{u}_m\}_{m=1}^{\infty}$ is also a bounded seq in $W^{1,p}(\mathbb{R}^n)$
$W^{1,p}(V)$

To prove compactness Th, it suffices

to show that $\{\bar{u}_m\}_{m=1}^{\infty}$ has a convergent subseq

$\{\bar{u}_{m_j}\}_{j=1}^{\infty}$ in $L^q(V)$ for $1 \le q < p^*$.

$$\left(\because \quad \underset{\substack{\| \\ \|u_{m_j} - \bar{u}\|_{L^q(\Omega)}}}{\|\bar{u}_{m_j} - \bar{u}\|_{L^q(\Omega)}} \le \|\bar{u}_{m_j} - \bar{u}\|_{L^q(V)} \quad (\Omega \subset\subset V) \right).$$

$\because \bar{u}_{m_j} = u_{m_j}$ in Ω

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Now we have $\{\bar{u}_m\}_{m=1}^{\infty}$ ~~where~~.

$$\sup_m \|\bar{u}_m\|_{W^{1,p}(V)} \leq C, \quad \text{supp}(\bar{u}_m) \subset\subset V$$

Consider $\bar{u}_m^{\varepsilon} = \eta_\varepsilon * \bar{u}_m$.

By Cor1, we have

$$\|\bar{u}_m - \bar{u}_m^{\varepsilon}\|_{L^1(V)} \leq \varepsilon C \|D\bar{u}_m\|_{L^p(V)}$$
$$\leq \varepsilon C \underline{\|\bar{u}_m\|_{W^{1,p}(V)}} \text{ uniformly bounded}$$
$$\leq \varepsilon C \leftarrow \text{indep of } m.$$

$$\Rightarrow \lim_{\varepsilon \to 0} \|\bar{u}_m - \bar{u}_m^{\varepsilon}\|_{L^1(V)} = 0 \text{ (indep of } m)$$

Recall that the interpolation inequality

$$1 \leq s \leq r \leq t, \quad \frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$$
$$\|f\|_{L^r(V)} \leq \|f\|^{\theta}_{L^s(V)} \|f\|^{1-\theta}_{L^t(V)}$$

Now
$$1 \leq q < p^*$$
$$\frac{1}{p^*} < \frac{1}{q} \leq 1$$
$$\frac{1}{q} = \theta \cdot 1 + \frac{(1-\theta)}{p^*}$$

$$\Rightarrow (*) \; \|\bar{u}_m^{\varepsilon} - \bar{u}_m\|_{L^q(V)} \leq \|\bar{u}_m^{\varepsilon} - \bar{u}_m\|^{\theta}_{L^1(V)} \cdot \|\bar{u}_m^{\varepsilon} - \bar{u}_m\|^{1-\theta}_{L^{p^*}(V)}$$

(By interpolation ineq.)

Recall that $\bar{u}_m \in W^{1,p}(V)$, $\text{supp}(\bar{u}_m) \subset\subset V$

$$\|\bar{u}_m\|_{L^{p^*}(V)} \leq C \|\bar{u}_m\|_{W^{1,p}(V)}$$
$$\|\bar{u}_m^{\varepsilon}\|_{L^{p^*}(V)} \leq \|\bar{u}_m\|_{L^{p^*}(V)}$$

(Exercise)

$$\Rightarrow \|\bar{u}_m - \bar{u}_m^{\varepsilon}\|_{L^{p^*}(V)} \leq \|\bar{u}_m\|_{L^{p^*}(V)} + \|\bar{u}_m^{\varepsilon}\|_{L^{p^*}(V)}$$
$$\leq 2\|\bar{u}_m\|_{L^{p^*}(V)}$$
$$\leq 2C \underline{\|\bar{u}_m\|}_{W^{1,p}(V)}$$
$$\leq C$$

From (*) $\Rightarrow \|\bar{u}_m^{\varepsilon} - \bar{u}_m\|_{L^q(V)} \leq \|\bar{u}_m^{\varepsilon} - \bar{u}_m\|^{\theta}_{L^1(V)} C^{1-\theta}$

Since $\lim_{\varepsilon \to 0} \|\bar{u}_m^{\varepsilon} - \bar{u}_m\|_{L^1(V)} = 0$ (uniformly in m)

$$\Rightarrow \lim_{\varepsilon \to 0} \|\bar{u}_m^{\varepsilon} - \bar{u}_m\|_{L^q(V)} = 0 \text{ (uniformly in } m)$$
$$\text{for } 1 \leq q < p^*$$

Since $\text{supp}(\bar{u}_m) \subset\subset V$, $\sup_m \|\bar{u}_m\|_{L^1(V)} < \infty$

$$\left(\text{b/c } \int_V |\bar{u}_m| \leq \left(\int_V |\bar{u}_m|^p\right)^{\frac{1}{p}} \cdot (\text{Vol}(V))^{\frac{p-1}{p}} \leq C \cdot \|\bar{u}_m\|_{W^{1,p}(V)} \right)$$

and $\text{supp}(\bar{u}_m^{\varepsilon}) \subset\subset V$ for ε small enough

$\Rightarrow$ By Lemma 2,

For each $\varepsilon > 0$, $\{\bar{u}_m^{\varepsilon}\}_{m=1}^{\infty}$ has a convergent subsequence in $L^q(V)$.

Now, fix $\delta > 0$,

We'll show that $\exists$ a subseq

$\{\bar{u}_{m_j}\}_{j=1}^{\infty}$, s.t $j, k \geq N$

$\Rightarrow \quad \|\bar{u}_{m_j} - \bar{u}_{m_k}\|_{L^q(V)} \leq \delta$

Recall that $\lim\limits_{\varepsilon \to 0} \|\bar{u}_m^\varepsilon - \bar{u}_m\|_{L^q(V)} = 0$

We can find ε small enough s.t

$\|\bar{u}_m^\varepsilon - \bar{u}_m\| < \frac{\delta}{3}$ for all m

Now for this ε, we can find a subseq

$\{\bar{u}_{m_j}^\varepsilon\}_{j=1}^{\infty}$ in $L^q(V)$ s.t

$\|\bar{u}_{m_j}^\varepsilon - \bar{u}_{m_k}^\varepsilon\|_{L^q(V)} < \frac{\delta}{3}$ for $j, k \geq N$

$$\Rightarrow \|\bar{u}_{m_j} - \bar{u}_{m_k}\|_{L^q(V)} \leq \|\bar{u}_{m_j} - \bar{u}_{m_j}^\varepsilon\|_{L^q(V)} + \|\bar{u}_{m_j}^\varepsilon - \bar{u}_{m_k}^\varepsilon\|_{L^q(V)} + \|\bar{u}_{m_k}^\varepsilon - \bar{u}_{m_k}\|_{L^q(V)}$$
$$< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta$$

Now let $\delta = 1, \frac{1}{2}, \cdots$ $(\delta \to 0)$

One can use "diagonal process" to

find a subseq s.t $\lim\limits_{j,k \to \infty} \|\bar{u}_{m_j} - \bar{u}_{m_k}\|_{L^q(V)} = 0$

b/c $L^q(V)$ is complete

$\Rightarrow \quad \{\bar{u}_{m_j}\}$ converges in $L^q(V)$ #.

We have proved that

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

where
- $1 \leq p < n$
- $1 \leq q < p^* = \frac{np}{n-p}$ $(> p)$
- Ω is bounded, $\partial\Omega$ is smooth.

Remark:

1° $W_0^{1,p}(\Omega) \subset\subset L^q(\Omega)$

even if $\partial\Omega$ is not C^1.

(Homework.)

2° $p \to n^-$, $p^* \to \infty$

$W^{1,p}(\Omega) \subset\subset L^p(\Omega)$.

(Discuss next time!)

$p^* > p$

$$\left(\text{b/c. } \frac{np}{n-p} - p = \frac{np - np + p^2}{n-p} = \frac{p^2}{n-p} > 0 \right)$$