Next, we want to establish Sobolev inequality for $W^{k,p}(\Omega)$.

We study the case for $W_0^{k,p}(\Omega)$.

Recall that

① $W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$ in $W^{k,p}(\Omega)$

So given any $u \in W_0^{k,p}(\Omega)$

$\exists \{u_m\}_{m=1}^{\infty}$ with $u_m \in C_0^\infty(\Omega)$

s.t. $\lim\limits_{k\to\infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$

② Suppose Ω is bounded

$1 \le q < p$, $u \in L^p(\Omega)$

$$\Rightarrow \quad \|u\|_{L^q(\Omega)} \le \left(Vol(\Omega)\right)^{\frac{1}{q}-\frac{1}{p}} \|u\|_{L^p(\Omega)}$$

pf: $\left(\int u^q \cdot 1\right) \le \left(\int (u^q)^{\frac{p}{q}}\right)^{\frac{q}{p}}$

$\left(\int_\Omega 1^{\cdot\cdot}\right)^{\frac{q}{p}-1}$

$$\left(\int u^q\right) \le \left(\int u^p\right)^{\frac{q}{p}} (Vol(\Omega))^{\frac{q}{p}-1}$$

$$\Rightarrow \quad \|u\|_{L^q(\Omega)} \le \left(Vol(\Omega)\right)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^p(\Omega)}$$

Th3 on p265

Assume Ω is a bounded, open subset of $\mathbb{R}^n$.

Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \le p < n$.

Then $\|u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{p^*}-\frac{1}{q}} \dfrac{p(n-1)}{n-p} \|Du\|_{L^p(\Omega)}$

for each $1 \le q \le p^*$

$|\Omega| = Vol(\Omega)$.