§5.6 Sobolev Inequalities

$$W^{1,p}(\Omega) = \{ u \mid u, |Du| \in L^p(\Omega) \}$$

$$W^{1,p}(\Omega) \overset{?}{\hookrightarrow} L^q(\Omega)$$

We want to see if

$$\|u\|_{L^q(\Omega)} \le C \|Du\|_{L^p(\Omega)}$$

where C depends only on p, n, Ω.

In the following, we'll prove that

$$\|u\|_{L^{\frac{np}{n-p}}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Motivation:

Find out the possible exponent where

$$(*) \quad \|u\|_{L^q(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)}$$

for $u \in C_0^\infty(\mathbb{R}^n)$

Suppose $(*)$ holds for u.

Let $u_\lambda(x) = u(\lambda x)$ when $\lambda > 0$

Since $u \in C_0^\infty(\mathbb{R}^n)$, $u_\lambda \in C_0^\infty(\mathbb{R}^n)$

So $\|u_\lambda\|_{L^q(\mathbb{R}^n)} \le C \|Du_\lambda\|_{L^p(\mathbb{R}^n)}$

Compute $\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \left(\int |u_\lambda(x)|^q dx \right)^{\frac{1}{q}}$

$$= \left(\int_{\mathbb{R}^n} |u(\lambda x)|^q dx \right)^{\frac{1}{q}} : \; y = \lambda x, \quad \frac{dy}{\lambda^n} = dx$$

$$= \left(\int_{\mathbb{R}^n} \frac{|u(y)|^q dy}{\lambda^n} \right)^{\frac{1}{q}}$$

$$= \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}$$

$$Du_\lambda(x) = D(u(\lambda x)) = (Du)(\lambda x) \cdot \lambda$$

$$\Rightarrow \|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |Du_\lambda(x)|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}^n} |(Du)(\lambda x)|^p \cdot \lambda^p dx \right)^{\frac{1}{p}} \quad \boxed{\begin{aligned} y &= \lambda x \\ dx &= \tfrac{1}{\lambda^n} dy \end{aligned}}$$

$$= \left(\int |Du(y)|^p \frac{\lambda^p}{\lambda^n} dy \right)^{\frac{1}{p}}$$

$$= \lambda^{\frac{p-n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

We have

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)}$$

$$\|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{\frac{p-n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} \leq C \|Du_\lambda\|_{L^p(\mathbb{R}^n)}$$

$$\Longleftrightarrow \quad \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \quad \|u\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{1-\frac{n}{p}+\frac{n}{q}} \|Du\|_{L^p(\mathbb{R}^n)}$$

If $1-\frac{n}{p}+\frac{n}{q} > 0$

then $\lim\limits_{\lambda\to\infty} \lambda^{1-\frac{n}{p}+\frac{n}{q}} = \infty$

If $1-\frac{n}{p}+\frac{n}{q} < 0$

then $\lim\limits_{\lambda\to 0^+} \lambda^{1-\frac{n}{p}+\frac{n}{q}} = \infty$

So $\|u\|_{L^q(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$

If $1-\frac{n}{p}+\frac{n}{q} = 0$

$$\Leftrightarrow \quad \frac{n}{q} = \frac{n}{p} - 1$$

$$\Leftrightarrow \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n} \left(= \frac{n-p}{np}\right)$$

$$\Leftrightarrow \quad q = \frac{np}{n-p}$$

Def: If $1 \leq p < \infty$

the Sobolev conjugate of p

is $p^* = \frac{np}{n-p}$. Note that

$$\frac{1}{p^*} + \frac{1}{n} = \frac{1}{p}.$$

Th2 (Gagliardo-Nirenberg Sobolev Inequality)

Assume $1 \leq p < n$, $\exists$ a constant C

depending only on p, n st

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_0^1(\mathbb{R}^n)$.

pf: First, prove the case when $p=1$

$$\Rightarrow \quad p^* = \frac{n}{n-1} \quad \left(p^* = \frac{np}{n-p}\right)$$

We want to show that

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq C \left(\int_{\mathbb{R}^n} |Du| dx\right)$$

$$\Leftrightarrow \left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx\right) \le C \left(\int_{\mathbb{R}^n} |Du|\right)^{\frac{n}{n-1}}$$

Since u has compact support,

$$u(x_1, \dots x_{i-1}, z_i, x_{i+1}, \dots x_n) = 0$$

when $\|z_i\|$ is large

$$\Rightarrow u(x) = \int_{-\infty}^{x_i} \frac{\partial u(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots x_n)}{\partial x_i}\, dy_i$$

$$\Rightarrow |u(x)| \le \int_{-\infty}^{\infty} \underbrace{\left|\frac{\partial u(x_1, \dots x_{i-1}, y_i, x_{i+1}, \dots, x_n)}{\partial x_i}\right|}_{\text{indep of } x_i} dy_i$$

$$\Rightarrow |u(x)|^{\frac{1}{n-1}} \le \left(\int_{-\infty}^{\infty} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots x_n)|\, dy_i\right)^{\frac{1}{n-1}}$$

$$\Rightarrow |u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(\dots, y_i, \dots)|\, dy_i\right)^{\frac{1}{n-1}}$$

$$\Rightarrow \int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\, dx_1$$

$$\le \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} |Du(y_1, x_2, \dots x_n)|\, dy_1\right)^{\frac{1}{n-1}}}_{\text{indep of } x_1} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du(\dots, y_i, \dots)|\, dy_i\right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{-\infty}^{\infty} |Du(y_1, x_2, \dots x_n)|\, dy_1\right)^{\frac{1}{n-1}} \cdot \int_{-\infty}^{\infty} \prod_{i=2}^{n} \boxed{\left(\int_{-\infty}^{\infty} |Du(\dots, y_i, \dots)|\, dy_i\right)^{\frac{1}{n-1}}} dx_1$$

$$\left(\int_{\mathbb{R}} \prod_{i=2}^{n} |f_i(z)|\, dz \le \prod_{i=2}^{n} \|f_i\|_{L^{n-1}(\mathbb{R})} \right. \qquad \|f_i\|_{L^{n-1}(\mathbb{R})} = \left(\int |f_i|^{n-1}\right)^{\frac{1}{n-1}}$$

$$\underbrace{\frac{1}{n-1} + \cdots + \frac{1}{n-1}}_{n-1 \text{ terms}} = 1$$

$$f_i \in L^{n-1}(\mathbb{R})$$

Holder Inequality

$$\le \left(\int_{-\infty}^{\infty} |Du(y_1, x_2, \dots x_n)|\, dy_1\right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du(\dots, y_i, \dots)|\, dy_i\, dx_1\right)^{\frac{1}{n-1}}$$

Holder