§5.7 Compactness

We have proven that

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

if $1 \le p < n$ and $1 \le q \le p^* = \frac{np}{n-p}$

$\begin{cases} \Omega \text{ is bounded} \\ \partial\Omega \text{ is } C^1. \end{cases}$

In the following, we'll prove that:

$$\underset{\text{cpt embedded}}{W^{1,p}(\Omega) \subset\subset L^q(\Omega)}$$

for $1 \le q < p^*$

Def: Let X and Y be Banach spaces, $X \subset Y$. We say X is compactly embedded in Y, written $X \subset\subset Y$,

provided

① $\|x\|_Y \le C \|x\|_X$ for $x \in X$ and some constant C indep of x

② each bounded sequence in X is precompact in Y.

(i.e. $\{x_m\}$ is a bounded seq in X $\exists$ subseq $\{x_{m_j}\}_{j=1}^\infty$ s.t. $\{x_{m_j}\}$ converges in Y

Note that this subsequence may not converge in X.)

Remark: $\overline{X} \subset Y$ if $X \subset\subset Y$

b/c $y \in \overline{X}$, $\exists$ $y_n \to y$ s.t. $\lim_{n\to\infty} \|y_n - y\|_X = 0$

$\|y_n - y\|_Y \le C \|y_n - y\|_X$

$\Rightarrow$ $\lim_{n\to\infty} \|y_n - y\|_Y = 0 \Rightarrow y \in Y$

Th1 on p272 (Rellich-Kondrachov Compactness Theorem)

Assume Ω is a bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1.

Suppose $1 \le p < n$. Then

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \le q < p^*$.

Pf: We have proven that

$$\|u\|_{L^q(\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$$

for $1 \le q \le p^*$.

To prove that $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for $1 \le q < p^*$, it suffices to show that if $\{u_m\}_{m=1}^\infty$ is a bounded seq in $W^{1,p}(\Omega)$

$\|u_m\|_{W^{1,p}(\Omega)} \le C$, $\exists$ a subseq $\{u_{m_j}\}$ which converges in $L^q(\Omega)$.

Let us prove the following Lemma first.

Lemma 1: if $u \in C^1(V) \cap W^{1,p}(V)$ and $\operatorname{supp} u \subset\subset V$ where V is bounded.

$\Rightarrow \|u-u^\varepsilon\|_{L^1(V)} \le \varepsilon\, C(n,p,V)\, \|Du\|_{L^p(\Omega)}$ for ε small enough.

pf: Recall that $u^\varepsilon = \eta_\varepsilon * u$.

$$u^\varepsilon(x) = \int_{B(0,1)} \eta(y)\, u(x-\varepsilon y)\, dy,$$

$$\int_{B(0,1)} \eta(y)\, dy = 1,$$

$$u(x) = \int_{B(0,1)} \eta(y)\, u(x)\, dy$$

(b/c $\operatorname{supp} u \subset\subset V$
$\Rightarrow \operatorname{supp} u^\varepsilon \subset\subset V$ if ε is small enough.)

$$|u^\varepsilon(x) - u(x)| = \left| \int_{B(0,1)} \eta(y)\, \underbrace{(u(x-\varepsilon y) - u(x))}\, dy \right|$$

$$= \left| \int_{B(0,1)} \eta(y) \left\{ \int_0^1 \left(\frac{d}{dt} u(x-\varepsilon t y) \right) dt \right\} dy \right|$$

$$\left(\int_0^1 Du(x-\varepsilon t y)\cdot(-\varepsilon y)\, dt \right)$$

b/c $\|y\| \le 1$

$$\le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 |Du(x-\varepsilon t y)|\, dt\, dy$$

$$\|u-u^\varepsilon\|_{L^1(V)} = \int_V |u(x) - u^\varepsilon(x)|\, dx$$

$$\le \varepsilon \int_V \int_{B(0,1)} \eta(y) \int_0^1 |Du(x-\varepsilon t y)|\, dt\, dy\, dx$$

$$= \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \underbrace{\int_V \|Du(x-\varepsilon t y)\|\, dx}_{=\ \|Du\|_{L^1(V)}}\, dt\, dy \qquad z = x-\varepsilon t y$$

$$= \varepsilon \|Du\|_{L^1(V)} \underbrace{\int_{B(0,1)} \eta(y)\, dy}_{=1}$$

$$= \varepsilon \int_V |Du(x)|\, dx$$

Hölder ineq, $\frac{1}{p} + \frac{p-1}{p} = 1$

$$\le \varepsilon \left(\int_V |Du|^p dx \right)^{\frac{1}{p}} \left(\int_V (1)^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}}$$

$$= \varepsilon \|Du\|_{L^p(V)} \underbrace{(\mathrm{Vol}(V))^{\frac{p-1}{p}}}_{C(n,p,V)}$$

$$\Rightarrow \|u-u^\varepsilon\|_{L^1(V)} \le \varepsilon\, C(n,p,V)\, \|Du\|_{L^p(V)}$$

Cor $u \in W^{1,p}(V)$, $\operatorname{supp} u \subset\subset V$

$$\Rightarrow \|u-u^\varepsilon\|_{L^1(V)} \le \varepsilon\, C(n,p,V)\, \|Du\|_{L^p(V)}$$

pf: $u \in W^{1,p}(V)$, $\operatorname{supp} u \subset\subset V$

$\exists\ u_m \in C_0^\infty(V)$ s.t.

$u_m \to u$ in $W^{1,p}(\Omega)$

ie $\quad u_m \to u$ in $L^p(V)$

and $Du_m \to Du$ in $L^p(V)$

b/c $\quad u_m \in C_0^\infty(V)$

From previous Lemma,

$(*)\quad \|u_m - u_m^\varepsilon\|_{L^1(V)} \leq \varepsilon\, C(V, n, p)\, \|Du_m\|_{L^p(V)}$

Note that $\quad u_m^\varepsilon \xrightarrow[m\to\infty]{} u^\varepsilon$ in $L_1(V)$

$$\left(\begin{aligned} &\int_V |u_m^\varepsilon(x) - u^\varepsilon(x)|\, dx \\ &\leq \int_V \int_{B(0,1)} \eta(y)\, |u_m(x-\varepsilon y) - u(x-\varepsilon y)|\, dy\, dx \end{aligned} \right.$$

$$\leq \int_V |u_m(x-\varepsilon y) - u(x-\varepsilon y)|\, dx$$

$$= \|u_m - u\|_{L^1(V)}$$

$$\leq C(n,p,V)\, \|u_m - u\|_{L^p(V)}$$

Hölder ineq

b/c $\quad \|u_m - u\|_{L^p(V)} \to 0$

$\Rightarrow \quad \|u_m^\varepsilon - u^\varepsilon\|_{L^1(V)} \to 0$ as $m \to \infty$

From $*$, take $m \to \infty$

$\Rightarrow \quad \|u - u^\varepsilon\|_{L^1(V)} \leq \varepsilon\, C(n,p,V)\, \|Du\|_{L^p(V)}$