Last time, we proved the Morrey's inequality for C^1 ftn i.e.

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

when $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$, $n < p \le \infty$, $\gamma = 1 - \frac{n}{p}$

Next, we'll use density Th to prove that

Th: Let Ω be a bounded, open subset of $\mathbb{R}^n$ and suppose $\partial\Omega$ is C^1. Assume $n < p \le \infty$ and $u \in W^{1,p}(\Omega)$. Then $\exists\, u^*$ s.t $u^* = u$ a.e. and $u^* \in C^{0,\gamma}(\bar\Omega)$ for $\gamma = 1 - \frac{p}{n}$ with the estimate $\|u^*\|_{C^{0,\gamma}(\bar\Omega)} \le C(n,p,\Omega)\|u\|_{W^{1,p}(\Omega)}$

pf: Since $\partial\Omega$ is C^1, by Th 1 on §5.3

We can find an extension $\bar u = Eu \in W^{1,p}(\mathbb{R}^n)$ s.t

- $\bar u = u$ in Ω
- $\bar u$ has compact support in V, where $\Omega \subset\subset V \subset\subset \mathbb{R}^n$
- $\|\bar u\|_{W^{1,p}(\mathbb{R}^n)} \le C(p,\Omega,V)\|u\|_{W^{1,p}(\Omega)}$

Since $\bar u \in W^{1,p}(\mathbb{R}^n)$ and it has cpt support

$\exists\, \{u_m\}_{m=1}^{\infty}$, $u_m \in C_0^\infty(\mathbb{R}^n)$ s.t $\operatorname{supp}(u_m) \subset\subset V$

$u_m \to \bar u$ in $W^{1,p}(\mathbb{R}^n)$

By Morrey inequality for $C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$

$\Rightarrow$ $\|u_m - u_l\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} \le C(n,p,\Omega)\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}$

b/c $u_m \to \bar u$ in $W^{1,p}(\mathbb{R}^n)$

$\Rightarrow$ $\{u_m\}$ is a cauchy seq in $W^{1,p}(\mathbb{R}^n)$

implies that $\{u_m\}$ is a cauchy seq in $C^{0,\gamma}(\mathbb{R}^n)$, $\gamma = 1 - \frac{n}{p}$.

Recall that $C^{0,\gamma}(\mathbb{R}^n)$ is a Banach space

$\Rightarrow$ $u_m \xrightarrow[m\to\infty]{} u^* \in C^{0,\gamma}(\mathbb{R}^n)$

Since $u_m \to \bar u$ in $W^{1,p}(\mathbb{R}^n)$, we know

$\bar u = u^*$ a.e. $\Rightarrow$ $u = u^*$ a.e in Ω

(b/c $\bar u = u$ in Ω)

Also, we have

$$\|u_m\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C(n,p,\Omega)\,\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$$

and $\lim_{m\to\infty} u_m = u^*$ in $C^{0,\gamma}(\mathbb{R}^n)$

and $\lim_{m\to\infty} u_m = \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$

Let $m\to\infty$

$\Rightarrow$ (*) $\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C(n,p,\Omega)\,\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}$

Note that

(**) $\left\{ \begin{array}{l} \|u^*\|_{C^{0,\gamma}(\bar{\Omega})} \le \|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \\ \text{and} \quad \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C_1(n,V,\Omega)\,\|u\|_{W^{1,p}(\Omega)} \end{array} \right.$

(V depends on Ω)

From (*), (**), we have

$$\|u^*\|_{C^{0,\gamma}(\bar{\Omega})} \le \tilde{C}(n,p,\Omega)\,\|u\|_{W^{1,p}(\Omega)}$$

#

So far, we proved that.

Ω bounded, open subset in $\mathbb{R}^n$ and $\partial\Omega$ is C^1. $u \in W^{1,p}(\Omega)$

Case 1: $1 \le p < n$

$\Rightarrow$ $u \in L^q(\Omega)$ for $1 \le q \le p^* = \frac{np}{n-p}$
$\|u\|_{L^q(\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$

Case 2: $n < p \le \infty$

$\Rightarrow$ $u \in C^{0,1-\frac{n}{p}}(\bar\Omega)$ and
$\|u\|_{C^{0,1-\frac{n}{p}}(\bar\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$

We can generalize this result to $u \in W^{k,p}(\Omega)$.

Th (General Sobolev inequalities)

Let Ω be a bounded, open subset of $\mathbb{R}^n$ with C^1 boundary.

Assume $u \in W^{k,p}(\Omega)$

(i) If $kp < n$

then $u \in L^q(\Omega)$ when

$$1 \le q \le \frac{pn}{n-kp} \left(= \frac{1}{\frac{1}{p} - \frac{k}{n}} \right).$$

We also have

$$\|u\|_{L^q(\Omega)} \le C \|u\|_{W^{k,p}(\Omega)}$$

(ii) If $kp > n$

then $u \in C^{k-[\frac{n}{p}]-1,\gamma}(\bar\Omega)$, where

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number} < 1, & \text{if } \frac{n}{p} \text{ is an integer} \end{cases}$$

We also have the estimate

$$\|u\|_{C^{k-[\frac{n}{p}]-1,\gamma}(\bar\Omega)} \le C(n,p,\Omega) \|u\|_{W^{k,p}(\Omega)}$$

pf: Sketch of the pf:

$D^\beta u \in W^{1,p}(\Omega)$ for $|\beta| \le k-1$

Remark: we have

$n > kp \Rightarrow W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$

$$1 \le q \le \frac{pn}{n-kp}$$

In fact, we'll prove that $W^{k,p}(\Omega) \subset\subset L^q(\Omega)$