(Ctn on the pf of Morrey's inequality)

Last time, we proved that

$$|u(x)| \le C(n,p)\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

$$\sup_{x\in\mathbb{R}^n} |u(x)| \le C(n,p)\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

Recall we have proved the following lemma.

Lemma: $$\fint_{B(x,r)} |u(x)-u(y)|\,dy \le C(n) \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy$$

This lemma implies that

$$\fint_{B(x,r)} |u(x)-u(y)|\,dy \le C(n,p)\, r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

(pf: We have

$$\fint_{B(x,r)} |u(x)-u(y)|\,dy$$

$$\le C(n) \left(\int_{B(x,r)} |Du(y)| \cdot \frac{1}{|x-y|^{n-1}}\,dy \right)$$

$$\left(\frac{1}{p} + \underbrace{\frac{p-1}{p}}_{=\frac{1}{q}} = 1 \right)$$

Hölder ineq

$$\le C(n) \left(\int_{B(x,r)} |Du(y)|^p dy \right)^{\frac{1}{p}} \left(\int_{B(x,r)} \left(\frac{1}{|x-y|^{n-1}} \right)^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}}$$

Compute $\displaystyle\int_{B(x,r)} |x-y|^{-(n-1)\frac{p}{p-1}}\,dy$

$$\overset{|x-y|=t}{=} \int_0^r t^{-\frac{(n-1)p}{p-1}} (n\alpha_n) t^{n-1}\,dt$$

$$= (n\alpha_n) \int_0^r t^{n-1-\frac{(n-1)p}{p-1}}\,dt$$

$$= (n\alpha_n) \left. \frac{t^{n-\frac{(n-1)p}{p-1}}}{n-\frac{(n-1)p}{p-1}} \right]_0^r \qquad \boxed{\begin{aligned} & n-\frac{(n-1)p}{p-1} \\ &= \frac{np-n-np+p}{p-1} \\ &= \frac{p-n}{p-1} > 0 \\ & \text{b/c } p>n \end{aligned}}$$

$$= n\alpha_n \frac{r^{\frac{p-n}{p-1}}}{\frac{p-n}{p-1}}$$

$$= C(n,p)\, r^{\frac{p-n}{p-1}}$$

$$\Rightarrow \fint_{B(x,r)} |u(x)-u(y)|\,dy$$

$$\le C(n,p)\, \|Du\|_{L^p(B(x,r))} \left(r^{\frac{p-n}{p-1}} \right)^{\frac{p-1}{p}}$$

$$= C(n,p)\, \|Du\|_{L^p(B(x,r))}\, r^{1-\frac{n}{p}}$$

Next, we'll use the estimate

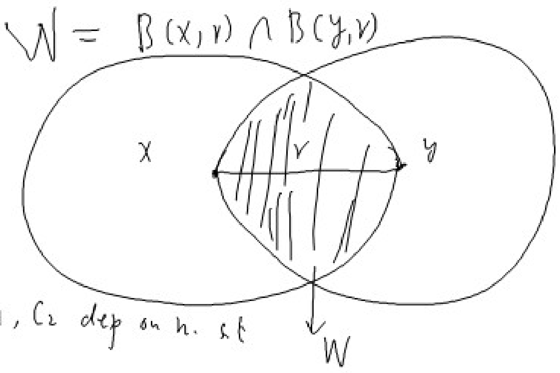
$$(*)\quad \fint_{B(x,r)} |u(x)-u(y)|\,dy \le C(n,p)\, r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

to prove that

$$\sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \le C\, \|Du\|_{L^p(\mathbb{R}^n)}$$

Choose any two point $x, y \in \mathbb{R}^n$ with $r = |x-y|$

Let $W = B(x,r) \cap B(y,r)$



$\exists\ C_1, C_2$ dep on n. st

$$C_1\big(\mathrm{Vol}(B(x,r))\big) \le \mathrm{Vol}(W) \le C_2\big(\mathrm{Vol}(B(x,r))\big)$$

$$|u(x)-u(y)| \le |u(x)-u(z)| + |u(z)-u(y)|$$

$$\int_W |u(x)-u(y)|\,dz \le \int_W \big(|u(x)-u(z)| + |u(z)-u(y)|\big)\,dz$$

$$|u(x)-u(y)|\,\mathrm{Vol}(W) \le \left(\int_W |u(x)-u(z)|\,dz\right) + \left(\int_W |u(z)-u(y)|\,dz\right)$$

$$\Rightarrow |u(x)-u(y)| \le \frac{\int_W |u(x)-u(z)|\,dz}{\mathrm{Vol}(W)} + \frac{\int_W |u(z)-u(y)|\,dz}{\mathrm{Vol}(W)}$$

We have

$$\frac{\int_W |u(x)-u(z)|\,dz}{\mathrm{Vol}(W)} \qquad (W \subset B(x,r))$$

$$\le \frac{C(n) \int_{B(x,r)} |u(x)-u(z)|\,dz}{\mathrm{Vol}(B(x,r))} \qquad \left(\frac{1}{\mathrm{Vol}(W)} \le \frac{1}{\underbrace{C_1}_{C(n)}\mathrm{Vol}(B(x,r))}\right)$$

$$= C(n) \underline{\fint_{B(x,r)} |u(x)-u(z)|\,dz}$$

$$\le C(n) \cdot \underbrace{C(n,p)\, r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}}_{\text{from } *}$$

$$\le C(n,p)\, |x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,r))}$$

Similarly, we also have

$$\frac{\int_W |u(y)-u(z)|\,dz}{\mathrm{Vol}(W)} \le C(n,p)\, |x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(B(y,r))}$$

$$\Rightarrow |u(x)-u(y)| \le C(n,p)\, |x-y|^{1-\frac{n}{p}} \left(\|Du\|_{L^p(B(x,r))} + \|Du\|_{L^p(B(y,r))}\right)$$

$$\le 2C(n,p)\, |x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \quad \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq C(n,p)\, \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \quad \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}} \leq C(n,p)\, \|Du\|_{L^p(\mathbb{R}^n)}$$

Combining this with

$$\sup_{x\in\mathbb{R}^n} |u(x)| \leq C(n,p)\, \|Du\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \|u\|_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |u(x)| + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{1-\frac{n}{p}}}$$

$$\leq C(n,p)\, \|Du\|_{L^p(\mathbb{R}^n)}$$

#

Remark:

The Key Lemma

$$\underbrace{\fint_{B(x,r)} \left| \underline{\underline{u(x)}} - \underline{u(y)} \right| dy}_{\geq \ \left| u(x) - \fint_{B(x,r)} u(y) \right|} \leq C(n,p)\, r^{1-\frac{n}{p}} \, \|Du\|_{L^p(B(x,r))}$$