Th4 (on p266) (Morrey's inequality)

Assume $n < p \le \infty$.

Then there exists a constant $C(p,n)$, such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$

where $\gamma := 1 - \frac{n}{p} > 0$.

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} |u(x)| + \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

Lemma: $$\fint_{B(x,r)} |u(y)-u(x)|\,dy \le C(n) \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy$$

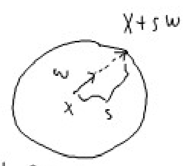
for $u \in C^1(B(x,2r))$

pf: Fix $w \in \partial B(0,1)$, $0 < s < r$.

$$|u(x+sw)-u(x)| = \left|\int_0^s \frac{d}{dt} u(x+tw)\,dt\right|$$

$$= \left|\int_0^s \nabla u(x+tw)\cdot w\,dt\right|, \quad |w|=1$$

$$\le \int_0^s |Du(x+tw)|\,dt$$



$$\Rightarrow \int_{\partial B(0,1)} |u(x+sw)-u(x)|\,dS(w)$$

$$\le \int_{\partial B(0,1)} \int_0^s |Du(x+tw)|\,dt\,dS(w)$$

$$= \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|\,dS(w)\,dt$$

Let $y = x+tw$

$\Rightarrow$ $dS(y) = t^{n-1}\,dS(w)$

$t w = y-x$

$t\,|w| = |y-x|$ (b/c $0 \le t \le s$, $|w|=1$)

$t = |y-x|$

$$= \int_0^s \int_{\partial B(x,t)} |Du(y)|\,\frac{dS(y)}{t^{n-1}}\,dt \quad (\text{use } t=|y-x|)$$

$$= \int_0^s \int_{\partial B(x,t)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dS(y)\,dt$$

$$= \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy$$

So we have

$$\int_{\partial B(0,1)} |u(x+sw)-u(x)|\,dS(w)$$

$$\le \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy$$

$$\le \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy \quad (\text{b/c } s<r)$$

$\Rightarrow$

$$\int_0^r \underbrace{\int_{\partial B(0,1)} |u(x+sw)-u(x)|\, dS(w)}\, \underline{s^{n-1}\, ds}$$

$$\leq \int_0^r \underbrace{\left(\int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy\right)}\, \underline{s^{n-1}\, ds}$$

$$\underset{y=x+sw}{\Longrightarrow} \int_{B(x,r)} |u(y)-u(x)|\, dy$$

$$\leq \left(\int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy\right)\left(\underbrace{\int_0^r s^{n-1}\, ds}_{\frac{r^n}{n}}\right)$$

$$\Rightarrow \frac{\int_{B(x,r)} |u(y)-u(x)|\, dy}{\alpha_n r^n} \leq \frac{1}{n\alpha_n}\left(\int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy\right)$$

$$\Rightarrow ⨍_{B(x,r)} |u(x)-u(y)|\, dy$$

$$\leq \underbrace{\left(\frac{1}{n\alpha_n}\right)}_{C(n)} \cdot \int_{B(x,r)} \frac{|Du(y)|\, dy}{|x-y|^{n-1}}$$

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pf of the Morrey's inequality

First we prove that

$$|u(x)| \leq C(p,n)\, \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

We have $\quad |u(x)| \leq |u(x)-u(y)| + |u(y)|$

$$\Rightarrow \int_{B(x,1)} |u(x)|\, dy \leq \int_{B(x,1)} |u(x)-u(y)|\, dy + \int_{B(x,1)} |u(y)|\, dy$$

$$|u(x)|\, |B(x,1)| \leq \int_{B(x,1)} |u(x)-u(y)|\, dy + \int_{B(x,1)} |u(y)|\, dy$$

$$\Rightarrow |u(x)| \leq ⨍_{B(x,1)} |u(x)-u(y)|\, dy + ⨍_{B(x,1)} |u(y)|\, dy$$

From previous lemma

$$⨍_{B(x,1)} |u(x)-u(y)|\, dy \leq C(n) \int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}}\, dy$$

$$\underset{\substack{\text{Hölder}\\ \text{ineq}}}{\leq} C(n) \cdot \left(\int_{B(x,1)} |Du(y)|^p\, dy\right)^{\frac{1}{p}} \cdot \underline{\left(\int_{B(x,1)} \left(\frac{1}{|x-y|^{n-1}}\right)^{\frac{p}{p-1}} dy\right)^{\frac{p-1}{p}}}$$

$$\left(\frac{1}{p} + \underbrace{\frac{p-1}{p}}_{=\frac{1}{q}} = 1\right)$$

$$\int_{B(x,1)} \frac{1}{|x-y|^{(n-1)\cdot\frac{p}{p-1}}}\, dy \underset{r=|x-y|}{=} \alpha_n \int_0^1 \frac{1}{r^{\frac{(n-1)p}{p-1}}}\, r^{n-1}\, dr$$

$$= \alpha_n \int_0^1 r^{n-1-\frac{(n-1)p}{p-1}}\, dr$$

$$= \alpha_n \left.\frac{r^{n-\frac{(n-1)p}{p-1}}}{n-\frac{(n-1)p}{p-1}}\right|_0^1 \qquad \left(\begin{aligned} &n-\frac{(n-1)p}{p-1} \\ &= \frac{np-n-np+p}{p-1} \\ &= \frac{p-n}{p-1} > 0 \end{aligned}\right)$$

$$= \frac{\alpha_n}{n-\frac{(n-1)p}{p-1}} = \frac{\alpha_n(p-1)}{(p-n)} = C(p,n) \qquad \text{b/c } p>n$$

$$\Rightarrow ⨍_{B(x,1)} |u(x)-u(y)|\, dy$$

$$\leq C(n,p)\left(\int_{B(x,1)} |Du(y)|^p\, dy\right)^{\frac{1}{p}}$$

$$\left(⨍_{B(x,1)} |u(y)|\, dy\right) \leq C(n,p)\left(\int_{B(x,1)} |u(y)|^p\, dy\right)^{\frac{1}{p}}$$

$(\Longrightarrow)$

$$|u(x)| \leq \fint_{B(x,1)} |u(x) - u(y)|\, dy + \fint_{B(x,1)} |u(y)|\, dy$$

$$\leq C_1(n,p) \left(\int_{B(x,1)} |Du(y)|^p \, dy \right)^{\frac{1}{p}} +$$

$$C_2(n,p) \left(\int_{B(x,1)} |u(y)|^p \, dy \right)^{\frac{1}{p}}$$

$$\leq C(n,p) \quad \|u\|_{W^{1,p}(\mathbb{R}^n)}$$