Remark:

Th Let $u, v \in L^1_{loc}(\Omega)$

Then $v = D^\alpha u$ (weak partial derivative)

$\Longleftrightarrow \exists$ a seq of $C^\infty(\Omega)$ $\{u_m\}_{m=1}^\infty$

s.t $\begin{array}{l} u_m \longrightarrow u \\ D^\alpha u_m \longrightarrow v \end{array}$ in $L^1_{loc}(\Omega)$

$\left(\begin{array}{l} \text{i.e } \Omega' \subset\subset \Omega \\ \lim_{m\to\infty} \int_{\Omega'} |u_m - u| = 0 \text{ and } \lim_{m\to\infty} \int_{\Omega'} |D^\alpha u_m - v| = 0 \end{array}\right)$

pf: $(\Rightarrow)$ Modify the pf of Th 2 (on p 25) and replace $\| \ \|_{W^{k,p}(\Omega)}$ by $\| \ \|_{L^1(\Omega)}$.

$(\Leftarrow)$

We want to prove that $D^\alpha u = v$

$$\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega v \phi$$

b/c $u_m \in C^\infty(\Omega)$

$$\Rightarrow \int_\Omega \underline{u_m D^\alpha \phi} = (-1)^{|\alpha|} \int_\Omega (D^\alpha u_m) \phi, \ \phi \in C_0^\infty(\Omega)$$

b/c $\phi, D^\alpha \phi$ has compact support

$$\Rightarrow \lim_{m\to\infty} \int_\Omega u_m D^\alpha \phi = \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega (D^\alpha u_m) \phi$$

$$\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega v \phi$$

b/c $\begin{array}{l} u_m \to u \\ D^\alpha u_m \to v \end{array}$ in $L^1_{loc}(\Omega)$

$\Rightarrow \begin{array}{l} u_m D^\alpha \phi \to u D^\alpha \phi \\ (D^\alpha u_m) \phi \to v \phi \end{array}$ in $L^1_{loc}(\Omega)$

$\Longrightarrow \quad D^\alpha u = v$

Last time, we proved that

If Ω is bounded, open and with C^1 boundary,

then given $u \in W^{1,p}(\Omega)$, $1 \le p < n$

we have $u \in L^q(\Omega)$ for $1 \le q \le p^*$

and $\|u\|_{L^q(\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$

$p^* = \underbrace{\frac{np}{n-p}}_{>1} > p$

Remark: If $p \to n^- \Rightarrow p^* \to \infty$

One "may" expect that from $\|u\|_{L^{p^*}(\Omega)} \le C(n,p,\Omega)\|u\|_{W^{1,p}(\Omega)}$

$\Downarrow p \to n^-$

to get $\|u\|_{L^\infty(\Omega)} \le C(n,\Omega) \|u\|_{W^{1,n}(\Omega)}$

But this not true from HW4 Problem 1

$\exists\ u \in W^{1,n}(B(0,1))$ st u is not bounded.

In the case when $p > n$, we have the following.

<u>Th5</u> on p269.

Let Ω be a bounded, open subset of $\mathbb{R}^n$ and suppose $\partial\Omega$ is C^1. Assume $n < p \le \infty$, and $u \in W^{1,p}(\Omega)$.

Then u has a version $u^* \in C^{0,\gamma}(\bar\Omega)$, for $\gamma = 1 - \frac{n}{p}$, with the estimate

$\|u^*\|_{C^{0,\gamma}(\bar\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$

Def: We say u^* is a version of a given ftn u if $u = u^*$ a.e.

Def (on p240, 5.1) Hölder Spaces

A ftn u is said to be Hölder continuous with exponent γ if $|u(x) - u(y)| \le C|x-y|^\gamma$ for all $x, y \in \Omega$.

Remark: If $\gamma = 1$, we called this is a Lipchitz cts ftn

2° A Hölder cts ftn
is uniformly cts.
$\Rightarrow u \in C^0(\Omega)$ + Hölder continuity
$\Rightarrow u \in C^0(\bar{\Omega})$

Def: (i) If $u : \Omega \longrightarrow \mathbb{R}$ is bounded
and continuous, we write
$$\|u\|_{C(\bar{\Omega})} = \sup_{x \in \Omega} |u(x)|$$

(ii) The γ^{th}-Hölder semi-norm of
$u : \Omega \longrightarrow \mathbb{R}$ is
$$[u]_{C^{0,\gamma}(\bar{\Omega})} = \sup_{\substack{x, y \in \Omega \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\gamma} \right\}$$
and the γ^{th}-Hölder norm is
$$\|u\|_{C^{0,\gamma}(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + [u]_{C^{0,\gamma}(\bar{\Omega})}$$

Def: The Hölder space $C^{k,\gamma}(\bar{\Omega})$
consists of all ftns $u \in C^k(\bar{\Omega})$
for which the norm
$$\|u\|_{C^{k,\gamma}(\bar{\Omega})} := \sum_{|\alpha| \le k} \|D^\alpha u\|_{C(\bar{\Omega})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{\Omega})}$$

Th $C^{k,\gamma}(\bar{\Omega})$ is a Banach space.

pf: It's not difficult to show
this is a normed linear space.
Given $\{u_m\}_{m=1}^\infty$ a cauchy seq in $C^{k,\gamma}(\bar{\Omega})$
We want to show it converges in $C^{k,\gamma}(\bar{\Omega})$
$\exists C$ s.t
b/c $\underset{|\alpha|=k}{[D^\alpha u_m]_{C^{0,\gamma}(\bar{\Omega})}} < C$

$$\Rightarrow \frac{|D^\alpha u_m(x) - D^\alpha u_m(y)|}{|x-y|^\gamma} \le C$$

$\Rightarrow |D^\alpha u_m(x) - D^\alpha u_m(y)| \le C\,|x-y|^\gamma$

$\Rightarrow \{D^\alpha_m u\}$ is equi—cts.

$\Rightarrow$ Also $\|D^\alpha u\|_{C(\Omega)} < \infty$

$\Rightarrow \{D^\alpha u_m\}$ is uniformly and equi-cts

$\Rightarrow \{D^\alpha u_m\}$ converge to Hölder cts ftn

$\Rightarrow$ Use diagonal process $u_m \to u$
in $C^{k,\gamma}(\bar{\Omega})$.