First, we want to state a Th about the approximation of $W^{1,p}(\Omega)$.

Th1 (p254)

Assume Ω is bounded (open) and $\partial\Omega$ is C^1.

Select a bounded open set V s.t $\Omega \subset\subset V$.

Then there exists a bounded linear operator. st

$$E: W^{1,p}(\Omega) \longrightarrow W^{1,p}(\mathbb{R}^n) \text{ s.t}$$

for each $u \in W^{1,p}(\Omega)$

(i) $Eu = u$ a.e. in Ω

(ii) Eu has compact support in V

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le \underset{\substack{\uparrow \\ \text{constant depending on } p, n, \Omega}}{C(n,p,\Omega)} \|u\|_{W^{1,p}(\Omega)}$

Def: We call Eu an extension of u to $\mathbb{R}^n$.

We'll use this Th to prove

Th 2 (Estimate for $W^{1,p}(\Omega)$ $1 \le p < n$)
↑ p265.

Let Ω be a bounded, open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1

Assume $1 \le p < n$ and $u \in W^{1,p}(\Omega)$

Then $u \in L^q(\Omega)$ for $1 \le q \le p^*$ with the estimate

$$\|u\|_{L^q(\Omega)} \le C \|u\|_{W^{1,p}(\Omega)}$$

when $C = C(n, p, \Omega)$ (constant).

pf: 1° It suffice to prove the case $q = p^*$.

For $1 \le q < p^*$

$$\|u\|_{L^q} \le C'(n,p,\Omega) \|u\|_{L^{p^*}(\Omega)}$$
$$\le \underbrace{C'(n,p,\Omega)\, C(n,p,\Omega)}_{\text{Const}(n,p,\Omega)} \|u\|_{W^{1,p}(\Omega)}$$

2° From previous Th, since $\Omega \subset\subset V \subset \mathbb{R}^n$

we have $\bar{u} = Eu \in W^{1,p}(\mathbb{R}^n)$

and $\bar{u}$ has compact support in V

$\bar{u} = u$ in Ω, $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$

Since $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ and $\bar{u}$ has compact support in V, we can find

$\{u_m\}_{m=1}^{\infty}$ s.t. $\lim_{m\to\infty} \|u_m - \bar{u}\|_{W^{1,p}(\mathbb{R}^n)} = 0$, i.e.

$$0 = \lim_{m\to\infty} \|u_m - \bar{u}\|_{L^p(\mathbb{R}^n)} + \sum_{|\alpha|=1} \|D^\alpha u_m - D^\alpha \bar{u}\|_{L^p(\mathbb{R}^n)}$$

where $u_m \in C_0^\infty(\mathbb{R}^n)$

(from Th 1 on p250)

$\to$ This implies that $\bar{u} \in W^{1,p}_{loc}(V)$

b/c $u_k - u_l \in C_0^\infty(\mathbb{R}^n)$

By Gagliardo-Nirenberg-Sobolev inequality (Th 1 on p263), we have

$$\|u_k - u_l\|_{L^{p^*}(\mathbb{R}^n)} \le \|Du_k - Du_l\|_{L^p(\mathbb{R}^n)}$$

Since $\{Du_m\}_{m=1}^{\infty}$ is a cauchy seq in $L^p(\mathbb{R}^n)$

$\Longrightarrow$ $\{u_m\}_{m=1}^{\infty}$ is a cauchy seq in $L^{p^*}(\mathbb{R}^n)$

b/c $L^{p^*}(\mathbb{R}^n)$ is complete and the limit is unique $\Rightarrow \lim_{m\to\infty} \|u_m - \bar{u}\|_{L^{p^*}(\mathbb{R}^n)} = 0$

$$\Longrightarrow \quad \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le \|Du_m\|_{L^p(\mathbb{R}^n)}$$

$$\Rightarrow \quad \lim_{m\to\infty} \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le \lim_{m\to\infty} \|Du_m\|_{L^p(\mathbb{R}^n)}$$

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le \|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

Now $\Omega \subset \mathbb{R}^n$ and $u \equiv \bar{u}$ in Ω.

$$\|u\|_{L^{p^*}(\Omega)} \le \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le \|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

$$\|u\|_{L^{p^*}(\Omega)} = \left(\int_\Omega |u|^{p^*}\right)^{\frac{1}{p^*}}, \qquad \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\bar{u}|^{p^*}\right)^{\frac{1}{p^*}}$$

$$= \left(\int_\Omega |u|^{p^*} + \underbrace{\int_{\mathbb{R}^n \setminus \Omega} |\bar{u}|^{p^*}}_{\ge 0}\right)^{\frac{1}{p^*}}$$

Also $\|D\bar{u}\|_{L^p(\mathbb{R}^n)} \le \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}$

$$\|D\bar{u}\|_{L^p(\mathbb{R}^n)} = \sum_{|\alpha|=1} \left(\int_\Omega |D^\alpha \bar{u}|^p\right)^{\frac{1}{p}}, \qquad \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} = \|\bar{u}\|_{L^p(\mathbb{R}^n)} + \sum_{|\alpha|=1} \|D^\alpha \bar{u}\|_{L^p(\mathbb{R}^n)}$$

$$= \|Eu\|_{W^{1,p}(\mathbb{R}^n)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$$

From our choice of $\bar{u}$

$$\Rightarrow \|u\|_{L^{p^*}(\Omega)} \le C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$$

#.