Recall

① Given $u \in W_0^{k,p}(\Omega)$.

$\exists \; u_m \in C_0^\infty(\Omega)$ s.t

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$$

② Suppose Ω is bounded

$1 \le q < p$ and $u \in L^p(\Omega)$

$$\Rightarrow \|u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q}-\frac{1}{p}} \|u\|_{L^p(\Omega)}$$

③ $\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du\|_{L^p(\mathbb{R}^n)}$

for $u \in C_0^1(\mathbb{R}^n)$

where $1 \le p < n$, $p^* = \frac{np}{n-p}$

$C = C(n,p)$

Th3 (on p2 of 5)

Assume Ω is bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \le p < n$.

Then $\|u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{p^*}-\frac{1}{q}} \frac{p(n-1)}{n-p} \|Du\|_{L^p(\Omega)}$

for $q \in [1, p^*]$

pf: We'll prove the case where $q = p^*$ first.

Given $u \in W_0^{1,p}(\Omega)$.

$\exists \; \{u_m\}_{m=1}^\infty$ s.t $\lim_{m\to\infty} \|u_m - u\|_{W^{1,p}(\Omega)} = 0$

$(u_m \in C_0^\infty(\Omega))$

In particular $\lim_{m\to\infty} \|u_m - u\|_{L^p(\Omega)} + \sum_{|\alpha|=1} \|D^\alpha u_m - Du\|_{L^p(\Omega)} = 0$

We can extend $u_m = 0$ in $\mathbb{R}^n \setminus \Omega$

Then $u_m \in C_0^\infty(\mathbb{R}^n)$

$u_k - u_l \in C_0^\infty(\mathbb{R}^n)$

$$\Rightarrow \|u_k - u_l\|_{L^{p^*}(\mathbb{R}^n)} \le \frac{p(n-1)}{n-p} \|Du_k - Du_l\|_{L^p(\mathbb{R}^n)}$$

By Th 1 on p2 of 3

(Gagliardo-Nirenberg-Sobolev ineq)

Recall that $u_k \in C_0^\infty(\Omega)$

$$\Rightarrow \|u_k - u_l\|_{L^{p^*}(\Omega)} \le \frac{p(n-1)}{n-p} \|Du_k - Du_l\|_{L^p(\Omega)}$$

b/c $\{Du_m\}_{m=1}^\infty$ is a Cauchy seq in $L^p(\Omega)$

$\Rightarrow \{u_m\}_{m=1}^\infty$ is a Cauchy seq in $L^{p^*}(\Omega)$

$\Rightarrow u_m \to u$ in $L^{p^*}(\Omega)$ (b/c $u_m \to u$ in $L^p(\Omega)$)

We have

$$\|u_m\|_{L^{p^*}(\Omega)} \leq \frac{p(n-1)}{n-p} \|Du_m\|_{L^p(\Omega)}$$

$$\left(\text{b/c} \quad u_m \in C_0^\infty(\Omega) \right)$$

Since $\lim_{m\to\infty} \|u_m - u\|_{L^{p^*}(\Omega)} = 0$

and $\lim_{m\to\infty} \|Du_m - Du\|_{L^p(\Omega)} = 0$

$$\Rightarrow \quad \lim_{m\to\infty} \|u_m\|_{L^{p^*}(\Omega)} \leq \lim_{m\to\infty} \frac{p(n-1)}{n-p} \|Du_m\|_{L^p(\Omega)}$$

$$\|u\|_{L^{p^*}(\Omega)} \leq \frac{p(n-1)}{n-p} \|Du\|_{L^p(\Omega)}$$

For $1 \leq q < p^*$, we have

$$\|u\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q} - \frac{1}{p^*}} \|u\|_{L^{p^*}(\Omega)}$$

$$\leq \underbrace{|\Omega|^{\frac{1}{q} - \frac{1}{p^*}} \frac{p(n-1)}{n-p}}_{C(n,p,\Omega)} \|Du\|_{L^p(\Omega)} \quad \#.$$

Thm 2. Let Ω be a bounded, open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(\Omega)$. Then $\|u\|_{L^{p^*}(\Omega)} \leq C(n,p,\Omega) \|u\|_{W^{1,p}(\Omega)}$.
