Th2 (on p251)

(Global approximation by smooth ftns)

Assume Ω is bounded.

Let $u \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$.

Then there exist ftns $\{u_m\}_{m=1}^{\infty}$

$u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that

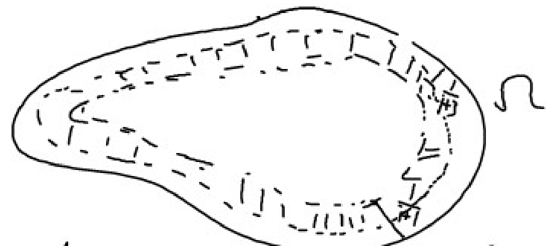
$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$$

Remark: $C^\infty \cap W^{k,p}(\Omega) \underset{\text{dense}}{\hookrightarrow} W^{k,p}(\Omega)$

pf: Let $U_i = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \frac{1}{i}\}$.

$\Rightarrow \Omega = \bigcup_{i=1}^{\infty} U_i$.

Let $V_i = U_{i+3} - \overline{U}_{i+1} = \{x \mid \frac{1}{i+3} < \operatorname{dist}(x, \partial\Omega) < \frac{1}{i+1}\}$

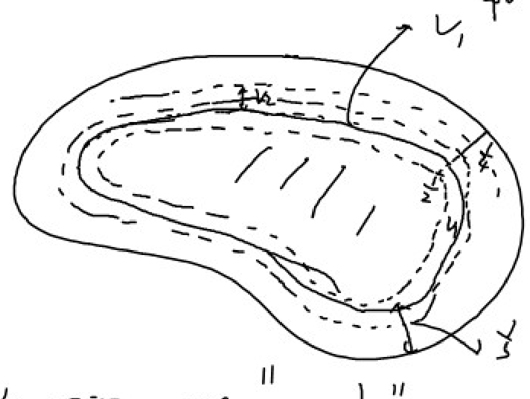


$V_1 = \{x \mid \frac{1}{4} < \operatorname{dist}(x, \partial\Omega) < \frac{1}{2}\}$.

$V_0 = \{x \mid \operatorname{dist}(x, \partial\Omega) > \frac{1}{3}\}$ ($\Rightarrow \overline{V_0} \subset\subset \Omega$)

Then $\Omega \subset \bigcup_{i=0}^{\infty} V_i$ and $\overline{V_i} \subset\subset \Omega$

cpt



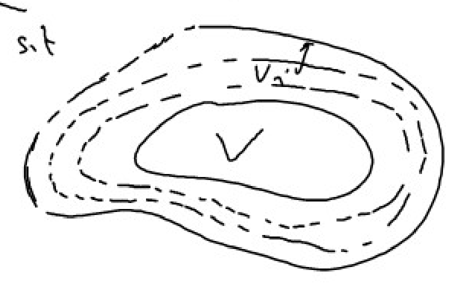
$V_1, V_2, \cdots$ are "annular" region

and getting close to the $\partial\Omega$.
as $i \to \infty$

In particular, given any open set

$V \subset\subset \Omega$

Then $V \cap V_i = \phi$ for $i \ge N$
$\exists N$ s.t



Now let $\{\phi_i\}_{i=0}^{\infty}$ be a smooth partition of unity subordinate to the open covering $\{V_i\}_{i=0}^{\infty}$, i.e.

$$\begin{cases} 0 \le \phi_i \le 1, \quad \phi_i \in C_0^\infty(V_i) \\ \underbrace{\sum_{i=0}^{\infty} \phi_i(x) = 1}_{\text{only a finite sum}} \quad \text{for } x \in \Omega \end{cases}$$

Now given $u \in W^{k,p}(\Omega)$

By Th 1 (iv) in § 5.2,

$\Rightarrow \quad \phi_i u \in W^{k,p}(\Omega)$

and $\operatorname{spt}(\phi_i u) \subset V_i \subset\subset \Omega$

By Th (we proved last time),

given any $\delta > 0$, $\exists\ \varepsilon_i$ small enough

s.t. $\quad u^i = \eta_{\varepsilon_i} * (\phi_i u)$ and

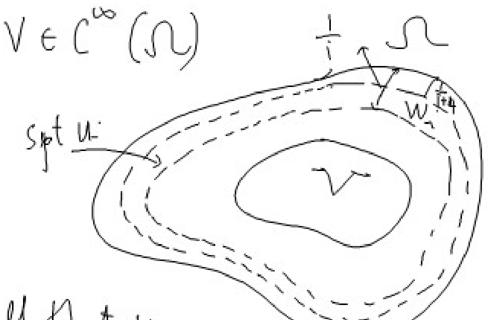
$$\| u^i - \phi_i u \|_{W^{k,p}(\Omega)} \le \frac{\delta}{2^{i+1}}$$

$$\operatorname{spt} u_i \subset W_i = U_{i+4} - \overline{U_i}$$

$$\left(\operatorname{sup}(\phi_i) \subset V_i = \underset{\cup}{U_{i+3}} - \overline{U_{i+1}} \right)$$

Let $\displaystyle V = \sum_{i=0}^{\infty} u^i = \sum_{i=0}^{\infty} \eta_{\varepsilon_i} * (\phi_i u)$

claim $V \in C^\infty(\Omega)$



Recall that $u_i = 0$ on $\Omega \setminus W_i$

Since $V \subset\subset \Omega$, $V \cap W_i \ne \phi$

for only a finite number of index i,

$\Rightarrow \quad x \in V \Rightarrow u_i(x) \ne 0$ for only a finite # of index

$x \in V$, $V(x) = \sum_i u_i(x)$ is only a finite sum.

$\Rightarrow$ V is smooth in $V \subset\subset \Omega$

$\Rightarrow$ V is smooth in Ω

Since $\displaystyle u = \sum_{i=0}^{\infty} \phi_i u \quad \left(\sum_{i=0}^{\infty} \phi_i = 1 \right)$

$$\Longrightarrow \| V - u \|_{W^{k,p}(V)} = \Big\| \sum_{i=0}^{\infty} (u_i - \phi_i u) \Big\|_{W^{k,p}(V)}$$

$$\le \sum_{i=0}^{\infty} \| u^i - \phi_i u \|_{W^{k,p}(\Omega)}$$

$$\le \sum_{i=0}^{\infty} \frac{\delta}{2^{i+1}} < \delta$$

$$\Longrightarrow \quad \|v-u\|_{W^{1,p}(\Omega)} \leq \delta$$

$$v \in C^{\infty}(\Omega) \cap W^{1,p}$$

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