In problem 4 (HW3).

$f'(0) = 0$

$\iff \int_\Omega \underbrace{\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}}_{\vec{X}} \cdot \nabla\phi = 0$ where $\phi \in C_0^\infty(\Omega)$

Recall $\vec{X} \cdot \nabla\phi$

$= \mathrm{div}(\vec{X}\phi) - \mathrm{div}(\vec{X})\phi$

(b/c $\mathrm{div}(\vec{X}\phi) = \mathrm{div}(\vec{X})\phi + \vec{X}\cdot\nabla\phi$)

$$0 = \int_\Omega \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\phi\right) - \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\phi$$

$$= \underbrace{\int_{\partial\Omega}\left(\frac{\nabla u\,\phi}{\sqrt{1+|\nabla u|^2}}\right)\cdot\nu}_{0 \text{ b/c } \phi \in C_0^\infty(\Omega)} - \int_\Omega \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\phi$$

$$\Rightarrow \int_\Omega \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\cdot\phi = 0$$

$$\Rightarrow \underbrace{\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0}_{\text{Eq for minimal graph.}}$$

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problem 2 (HW3)

Show that the Green's ftn $G(x,y) > 0$ in Ω.

pf: Recall that if u is harmonic in Ω (bounded) and $u \in C^0(\bar\Omega)$ then either

$$\inf_{\partial\Omega} u < u < \sup_{\partial\Omega} u$$

or $u \equiv$ constant.

Recall that

$$G(x,y) = \Phi(x,y) - \underline{\underline{\phi^x(y)}}$$

and $\Delta_y \phi^x(y) = 0$

$\phi^x(y)\big|_{\partial\Omega} = \Phi(x,y)$

Let fix x $u(z) = \Phi(x,z) - \phi^x(z)$

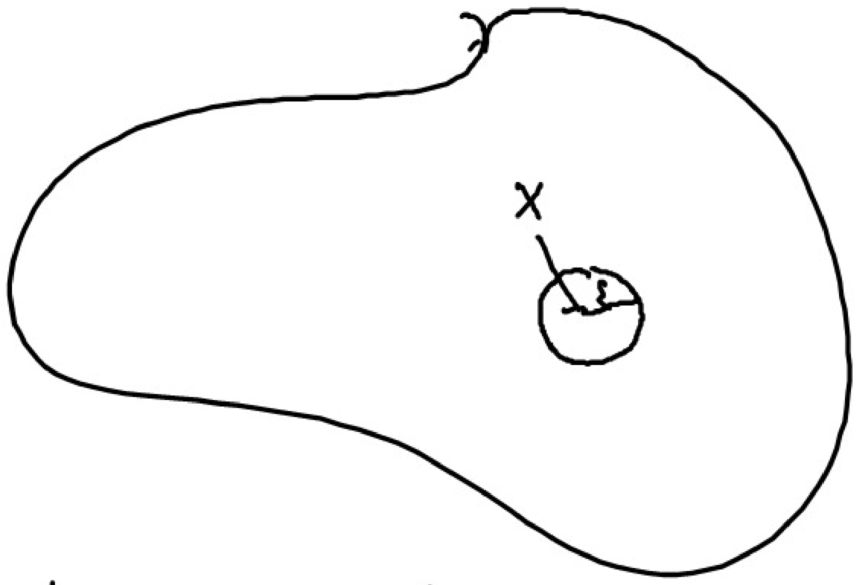
Then $\Delta u = 0$ if $z \neq x$

$\Delta_z \phi^x(z) = 0$ for all z.

$\phi^x(z)$ is a bounded ftn in Ω

$\lim_{z\to x} \Phi(x,z) = \infty$ (depends only on $\|x-z\|$)

$\Rightarrow \lim_{z\to x} u(z) = \lim_{z\to x} \Phi(x,z) - \underbrace{\phi^x(z)}_{\text{bounded}} = \infty$



So we can find ε small enough s.t

$$u(z) > 0 \text{ on } \partial B(x,\varepsilon)$$

Recall that $\underset{\Phi(x,z) - \phi^x(z)}{\underset{\parallel}{u(z)}} = 0$ on $\partial\Omega$.

So
$$\begin{cases} \Delta_z u = 0 & \text{in } \underline{\Omega \setminus B(x,\varepsilon)} \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{on } \partial B(x,\varepsilon) \end{cases}$$

$\Rightarrow$ $0 \le u$ on $\partial\left(\Omega \setminus B(x,\varepsilon)\right)$

and $u \not\equiv$ constant

$\Rightarrow$ By strong maximum principle

$$u > 0 \text{ in } \Omega \setminus B(x,\varepsilon)$$

$\Rightarrow$ $G(x,z) > 0$ for $z \in \Omega \setminus B(x,\varepsilon)$

This is true for all ε small enough

$$G(x,z) > 0 \text{ for all } z \in \Omega \setminus \{x\}.$$

#.

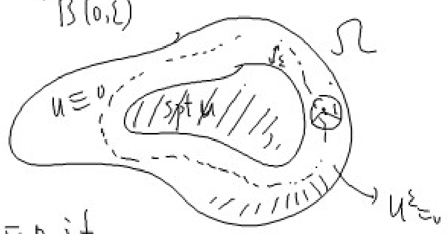
Last time, we proved that
if $u \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$
Let $V \subset\subset \Omega$.
Then $\lim\limits_{\varepsilon \to 0} \|u^\varepsilon - u\|_{W^{k,p}(V)} = 0$

$u^\varepsilon \in C^\infty(\Omega_\varepsilon)$.

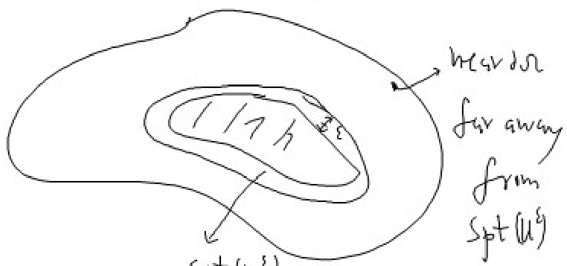
Remark: If $u \in W^{k,p}(\Omega)$ and
u has compact support in Ω
then $\lim\limits_{\varepsilon \to 0} \|u^\varepsilon - u\|_{W^{k,p}(\Omega)} = 0$

$u^\varepsilon \in C_0^\infty(\Omega)$ for ε small enough.

Pf: $u^\varepsilon(x) = \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\, u(x-y)\, dy$

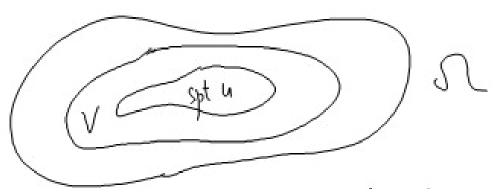


$u(x-y) = 0$ if $x-y \in \Omega \setminus (\operatorname{spt} u)$
So $u^\varepsilon(x) = 0$ if ε is small enough
and x is close to the boundary of Ω.
$\Rightarrow$ u^ε has compact support in Ω
for ε small enough



$\operatorname{spt}(u^\varepsilon)$
is "ε" distance from
spt u.

Choose $V \subset\subset \Omega$
and $\operatorname{spt} u \subset\subset V$



By previous Th $\|u^\varepsilon - u\|_{W^{k,p}(V)} = 0$ as $\varepsilon \to 0$

Also u^ε has compact support in V
if ε is small enough.
Since u^ε and u has compact support in V
$\Rightarrow$ $\|u^\varepsilon - u\|_{W^{k,p}(V)} = \|u^\varepsilon - u\|_{W^{k,p}(\Omega)}$ ($u^\varepsilon = u = 0$ on $\Omega \setminus V$)
So $u^\varepsilon \in C_0^\infty(\Omega)$ and
$\lim\limits_{\varepsilon \to 0} \|u^\varepsilon - u\|_{W^{k,p}(\Omega)} = 0$

Th (Partition of Unity)

Let Ω be an open set and

$$\Omega \subset \bigcup_{i=1}^{\infty} V_i$$ when V_i is open

and $\overline{V_i}$ is compact in Ω.

$(V_i \subset\subset \Omega)$

Then there exists a seq of ftn $\{\phi_i\}_{i=1}^{\infty}$ s.t

(i) $0 \le \phi_i \le 1$ and $\phi_i \in C_0^{\infty}(V_i)$

(ii) For each $x \in \Omega$,
there are only finitely many i
such that $\phi_i(x) \neq 0$

(iii) $\sum_{i=1}^{\infty} \phi_i(x) = 1$ for all $x \in \Omega$

(from (ii) this is only a finite sum)

Let $u \in W^{k,p}(\Omega)$

Th: $\exists\, u_m \in C^{\infty}(\Omega) \cap W^{k,p}(\Omega)$ s.t

$$\lim_{m \to \infty} \| u_m - u \|_{W^{k,p}(\Omega)} = 0$$