Problem 2 from HW2.

A $C^0(\Omega)$ ftn u is subharmonic in Ω if for every ball $B \subset\subset \Omega$ and every function h harmonic in B satisfying $u \le h$ on ∂B

$\Rightarrow \quad u \le h$ in B

Prove that a $C^0(\Omega)$ subharmonic ftn satisfying the strong maximum principle.

pf: We want to show that if u achieves an interior maximum

$\Rightarrow u \equiv$ constant in Ω

Suppose $\underline{x_0}$ achieves its interior maximum

$\Rightarrow$ We want to show that

$u \equiv$ constant on $B(x_0, r) \subset\subset \Omega$

$$\Rightarrow \left(\begin{array}{l} u \equiv \text{const in } \Omega \\ u(x_0) = \underset{\sup_\Omega u}{\overset{\shortparallel}{M}} \Rightarrow u^{-1}(M) \text{ is open and closed} \end{array} \right)$$

If $u \not\equiv$ const in $B(x_0, r)$

then we can find a r_0 s.t.

$$u\big|_{\partial B(x_0, r_0)} \not\equiv \text{constant}.$$

Consider h to be the harmonic ftn

when $\begin{cases} \Delta h = 0 & \text{in } B(x_0, r_0) \\ h\big|_{\partial B(x_0, r_0)} = u & (c\!b) \end{cases}$

In particular, $u \le h$ in $B(x_0, r_0)$

(b/c u is $C^0(\Omega)$ subharmonic ftn)

$\Rightarrow \quad u(x_0) \le h(x_0)$ *

Since h is harmon

$\Rightarrow$ (maximum principle) $\quad \max\limits_{\partial B(x_0, r_0)} h(x) = \max\limits_{B(x_0, r_0)} h$

$$\Rightarrow \quad ** \quad h(x_0) \le \max_{B(x_0, r_0)} h = \cancel{\max_{\partial B(x_0, r_0)} h(x)} = \max_{\partial B(x_0, r_0)} u(x) \le u(x_0)$$

b/c $u(x_0) = \sup\limits_{x \in \Omega} u$

From * and **

$$\Rightarrow \quad u(x_0) = h(x_0) = \max_{B(x_0, r_0)} h$$

b/c h is harmonic

By strong maximum principle $\Rightarrow h \equiv$ constant in $B(x_0, r_0)$

$$\Rightarrow \quad h\big|_{\partial B(x_0, r_0)} = u\big|_{\partial B(x_0, r_0)} \equiv \text{constant}$$

$\rightarrow\leftarrow$ we assume $\not\equiv$ constant.

§5.3 Approximation

§5.3.1 Interior approximation by smooth ftns.

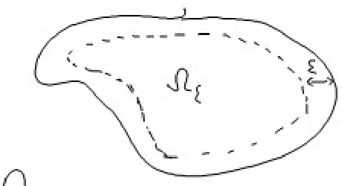
Goal: Given $u \in W^{k,p}(\Omega)$.

Then $\exists \{u_m\}_{m=1}^{\infty} \in C^{\infty}(\Omega) \cap W^{k,p}(\Omega)$

s.t $\|u_m - u\|_{W^{k,p}(\Omega)} \longrightarrow 0$

Recall $\Omega \subset \mathbb{R}^n$ open (bounded). $\varepsilon > 0$

① $\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$



$\Omega_\varepsilon \subset\subset \Omega$

② $\eta(x)$ is a radial ftn with

(a) $\int_{\mathbb{R}^n} \eta(x) = 1$

(b) $\eta(x) \geq 0$

(c) $\eta(x) > 0$ when $|x| < 1$

(d) $\eta(x) = 0$ when $|x| \geq 1$

Let $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$.

Then $\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx \equiv 1$,

$\eta_\varepsilon(x) = 0$ when $|x| \geq \varepsilon$

$\eta_\varepsilon(x) > 0$ when $|x| < \varepsilon$

Def: If $f : \Omega \to \mathbb{R}$ and $f \in L^1_{loc}(\Omega)$

define $f^\varepsilon(x) = (\eta_\varepsilon * f)(x)$ in Ω_ε

$\left(= \int_\Omega \eta_\varepsilon(x-y) f(y)\,dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y)\,dy \right.$

$(x \in \Omega_\varepsilon,\ y \in B(0,\varepsilon) \Rightarrow x-y \in \Omega)$

Properties of mollification.

(i) $f^\varepsilon \in C^\infty(\Omega_\varepsilon)$

(ii) $f^\varepsilon \longrightarrow f$ a.e as $\varepsilon \to 0$

(iii) If $f \in C^0(\Omega)$ then $f^\varepsilon \to f$ uniformly on cpt subsets of Ω.

(iv) If $1 \leq p < \infty$ and $f \in L^p_{loc}(\Omega)$

$\Longrightarrow f^\varepsilon \to f$ in $L^p_{loc}(\Omega)$

Th Assume $u \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$ and set $u^\epsilon = \eta_\epsilon * u$ in Ω_ϵ

Then (i) $u^\epsilon \in C^\infty(\Omega_\epsilon)$

(ii) $u^\epsilon \longrightarrow u$ in $W^{k,p}_{loc}(\Omega)$ as $\epsilon \to 0$

pf: 1° (i) follows from the property of mollification.

2°

$$u_\epsilon(x) = \int_\Omega \eta_\epsilon(x-y)\, u(y)\, dy$$

$$\left(= \int_{B(0,\epsilon)} \eta_\epsilon(y)\, u(x-y)\, dy \right)$$

$$D^\alpha u^\epsilon(x) = \int D^\alpha_x(\eta_\epsilon(x-y))\, u(y)\, dy$$

Since η_ϵ is radial symmetry

$$\Rightarrow D^\alpha_y \eta_\epsilon(x-y) = (-1)^{|\alpha|} D^\alpha_x \eta_\epsilon(x-y)$$

$$\Rightarrow D^\alpha u^\epsilon(x) = (-1)^{|\alpha|} \int D^\alpha_y (\eta_\epsilon(x-y))\, \underset{C_0^\infty(\Omega)}{u(y)}\, dy$$

$$\underset{\text{integration by parts}}{\overset{u \in W^{k,p}}{=}} (-1)^{|\alpha|} \cdot (-1)^{|\alpha|} \int_\Omega D^\alpha u(y)\, \eta_\epsilon(x-y)\, dy$$

$$= \int_\Omega \eta_\epsilon(x-y)\, D^\alpha u(y)\, dy$$

$$= (\underset{L^p(\Omega)}{D^\alpha u})^\epsilon$$

$$D^\alpha(\underset{\text{Smooth}}{u^\epsilon}) = (\underset{L^p(\Omega)}{D^\alpha u})^\epsilon$$

Let $V \subset\subset \Omega$.

Then $D^\alpha(u^\epsilon) = (D^\alpha u)^\epsilon \longrightarrow D^\alpha u$ in $L^p(V)$

$$\|u^\epsilon - u\|_{W^{k,p}(V)} = \sum_{|\alpha| \le k} \|D^\alpha(u_\epsilon) - D^\alpha(u)\|_{L^p(V)} \xrightarrow{\epsilon \to 0} 0$$

$u^\epsilon \to u$ in $W^{k,p}(V)$ as $\epsilon \to 0$

$\Rightarrow u^\epsilon \to u$ in $W^{k,p}_{loc}(\Omega)$