Ex. Let $\Omega = B(0,1) \subset \mathbb{R}^n$

and $u(x) = |x|^{-\delta}$, $\delta > 0$, $x \in \Omega$, $x \neq 0$

Then ① $u \in L^1(\Omega)$ if $n > \delta$

② $u \in W^1(\Omega)$ if $n > \delta + 1$

③ $u \in W^{1,p}(\Omega)$ iff $\delta < \frac{n-p}{p}$ $\Updownarrow$ $p(\delta+1) < n$

pf. Consider $\displaystyle\int_{\Omega \setminus B(0,\varepsilon)} |u(x)|\, dx$

$$= \int_{\Omega \setminus B(0,\varepsilon)} \frac{1}{|x|^\delta}\, \underline{dx}$$

$$= \int_\varepsilon^1 \frac{1}{r^\delta} \underline{(n\alpha_n) r^{n-1} dr}$$

$$= n\alpha_n \int_\varepsilon^1 r^{n-\delta-1}\, dr$$

$$= n\alpha_n \left. \frac{r^{n-\delta}}{n-\delta} \right]_\varepsilon^1 \quad \text{if } n-\delta \neq 0$$

$$= n\alpha_n \left(\frac{1}{n-\delta} - \frac{\varepsilon^{n-\delta}}{n-\delta} \right)$$

$\lim_{\varepsilon \to 0} \varepsilon^{n-\delta} = 0$ if $n-\delta > 0 \Rightarrow \int_\Omega |u(x)| < \infty$

$\lim_{\varepsilon \to 0} \varepsilon^{n-p} = \infty$ if $n-\delta < 0$

if $n-\delta = 0 \Rightarrow \displaystyle\int_{\Omega \setminus B(0,\varepsilon)} |u(x)| = n\alpha_n(-\ln\varepsilon) \downarrow \infty$ as $\varepsilon \to 0$

$\Longrightarrow u \in L^1(\Omega)$ iff $n-\delta > 0$

Claim: $u \in W^1(\Omega)$ if $n > \delta + 1$

$$\frac{\partial}{\partial x_i} u = \frac{\partial}{\partial x_i}\left(|x|^{-\delta} \right)$$

$$= -\delta |x|^{-\delta-1} \cdot \frac{x_i}{|x|} \quad \left(\frac{\partial}{\partial x_i}|x| = \frac{x_i}{|x|} \right).$$

$$= -\delta |x|^{-\delta-2} x_i$$

$$\Rightarrow |Du|^2 = \delta^2 |x|^{-2\delta-4} |x|^2 = \delta^2 |x|^{-2\delta-2}$$

$$\Rightarrow |Du| = \frac{\delta}{|x|^{\delta+1}}$$

To show that $u \in W^1(\Omega)$, we need to prove that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} = (-1) \int_\Omega \frac{\partial u}{\partial x_i} \phi$$

Also the weak derivative of u

$$\underbrace{\frac{\partial u}{\partial x}}_{W.D} = \frac{\partial u(x)}{\partial x} \text{ defined when } x \neq 0$$

Consider $\phi \in C_0^\infty(\Omega)$ and for $\varepsilon > 0$

$$(A) \int_{\Omega \setminus B(0,\varepsilon)} u \frac{\partial \phi}{\partial x_i} dx = -\int_{\Omega \setminus B(0,\varepsilon)} \frac{\partial u}{\partial x_i} \phi \, dx + \int_{\partial B(0,\varepsilon)} u \phi v^i ds$$

($\because \phi = 0$ on $\partial\Omega$)

$$\left| \int_{\partial B(0,\varepsilon)} u \phi v^i ds \right| \leq \|\phi\|_\infty \int_{\partial B(0,\varepsilon)} |u| \, ds$$

$v^i =$ i-th component of the unit normal vectors
$|v^i| \leq 1$

$$\int_{\partial B(0,\varepsilon)} |u| \, ds \underset{u = |x|^{-\delta}}{=} \frac{n \alpha_n \varepsilon^{n-1}}{\varepsilon^\delta} = n \alpha_n \varepsilon^{n-1-\delta}$$

If $n-1-\delta > 0$ then $\lim_{\varepsilon \to 0} n \alpha_n \varepsilon^{n-1-\delta} = 0$

$\Rightarrow \lim_{\varepsilon \to 0} \int_{\partial B(0,\varepsilon)} u \phi v^i = 0$, Let $\varepsilon \to 0$ in $*$

$$\Rightarrow \int_\Omega u \frac{\partial \phi}{\partial x_i} = (-1) \int_\Omega \frac{\partial u}{\partial x} \phi$$

$\Rightarrow u \in W^1(\Omega)$ if $n-1-\delta > 0$

$u \in W^{1,p}(\Omega)$ iff $\delta < \frac{n-p}{p}$

$D^i u = \frac{\partial u}{\partial x}$ if $x \neq 0$

We want to show that $\int |D^i u|^p < \infty$ if $\delta < \frac{n-p}{p}$

It suffices to prove that

$$\int_\Omega |Du|^p < \infty$$

Consider $\int_{\Omega \setminus B(0,\varepsilon)} |Du|^p \, dx$

$$= \int_{\Omega \setminus B(0,\varepsilon)} \left(\frac{\delta}{|x|^{\delta+1}} \right)^p dx$$

$$= n \alpha_n \int_\varepsilon^1 \delta^p r^{-p(\delta+1)} \cdot r^{n-1} dr$$

$$= n \alpha_n \delta^p \int_\varepsilon^1 r^{n-p(\delta+1)-1} dx$$

$$= n \alpha_n \delta^p \left. \frac{r^{n-p(\delta+1)}}{n-p(\delta+1)} \right]_\varepsilon^1$$

$\Rightarrow \lim_{\varepsilon \to 0} \int_{\Omega \setminus B(0,\varepsilon)} |Du|^p dx < \infty$ if $n - p(\delta+1) > 0$

$\Rightarrow u \in W^{1,p}$ if $n - p(\delta+1) > 0$

Ex. Let $\{r_k\}_{k=1}^{\infty}$ be a

countable dense subset in $B(0,1)$

Let $w(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\delta}$

where $\delta < \frac{n-p}{p}$

Then $w \in W^{1,p}$ and

w is unbounded on

any open set in $B(0,1)$

pf: $v_k = |x - r_k|^{-\delta} \in W^{1,p}$

from previous example

b/c $W^{1,p}$ is a Banach space

$\Rightarrow u_k = \sum_{l=1}^{k} \frac{1}{2^l} \cdot |x - r_l|^{-\delta} \in W^{1,p}$

$k > m$

$\Rightarrow \| u_k - u_m \|_{W^{1,p}} \le \sum_{l=m+1}^{k} \frac{1}{2^l} \| |x - r_l|^{-\delta} \|_{W^{1,p}}$

$\le C \cdot \sum_{l=m+1}^{k} \frac{1}{2^l}$

$\Rightarrow u_k$ is a Cauchy seq.

$\Rightarrow \lim_{k \to \infty} u_k$ converges in $W^{1,p}$

$\| |x - r_l|^{-\delta} \|_{W^{1,p}} \le C$

indep of l

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\delta} \in W^{1,p}$

if $\frac{n-p}{p} > \delta$