We want to prove that

$$D^{\alpha}(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^{\beta}\zeta \cdot D^{\alpha-\beta} u$$

where $\zeta \in C_0^{\infty}(\Omega)$ and $u \in W^{k,p}(\Omega)$.

Prove this by induction on $|\alpha|$.

If $|\alpha|=1$,

$$\int (\zeta u) D^{\alpha}\phi \underset{\zeta,\phi \in C_0^{\infty}(\Omega)}{=\!=} \int u (\underbrace{\zeta D^{\alpha}\phi}_{\text{when } |\alpha|=1})$$

$$\underset{\text{product rule}}{=\!=} \int u \left[\underbrace{D^{\alpha}(\zeta\phi) - (D^{\alpha}\zeta)\phi}_{} \right]$$

$$= \int u D^{\alpha}(\zeta\phi) - \int (u D^{\alpha}\zeta)\phi$$

$$\overset{\zeta\cdot\phi \in C_0^{\infty}(\Omega)}{\underset{\text{def of weak p.D.}}{=\!=}} (-1)^{|\alpha|}\int (D^{\alpha}u)(\zeta\underline{\underline{\phi}}) - \int (u D^{\alpha}\zeta)\underline{\underline{\phi}}$$

$$\underset{|\alpha|=1}{=} (-1)^{|\alpha|}\int (D^{\alpha}u\, \zeta + u D^{\alpha}\zeta)\phi$$

$$\Longrightarrow D^{\alpha}(u\zeta) = D^{\alpha}u\,\zeta + u D^{\alpha}\zeta$$

(weak p.D of ζu exists)

So Leibniz's formula holds when $|\alpha|=1$

Assume the L's formula holds when $|\alpha|=K$

Now $|\alpha|=K+1$

We can find β and γ s.t.

$\alpha = \beta+\gamma$ with $|\beta|=K$, $|\gamma|=1$

Consider $\int (\zeta u) D^{\alpha}\phi$

$$= \int (\zeta u) D^{\beta}\cdot(\underbrace{D^{\gamma}\phi}_{\substack{\uparrow \\ \text{test ftn in } C_0^{\infty}(\Omega)}})$$

$$\underset{\substack{\uparrow \\ \text{By induction} \\ |\beta|=K}}{=} (-1)^{|\beta|}\int D^{\beta}(\zeta u)\cdot(D^{\gamma}\phi)$$

$$= (-1)^{|\beta|}\int \left[\sum_{\sigma \le \beta}\binom{\beta}{\sigma} D^{\sigma}\zeta\, D^{\beta-\sigma}u\right] D^{\gamma}\phi$$

(By induction the w.p.D of $D^{\sigma}\zeta \cdot D^{\beta-\sigma}u$ exists

$\overset{\uparrow}{C_0^{\infty}(\Omega)}$ $|\beta-\sigma| \le K$

Also $W^{k,p}$ is a vector space

$\Rightarrow$ Weak p.D of $\sum_{\sigma\le\beta}\binom{\beta}{\sigma}D^{\sigma}\zeta D^{\beta-\sigma}u$ exists.

$$= (-1)^{|\gamma|+|\beta|}\int \left[\sum_{\sigma\le\beta}\binom{\beta}{\sigma} D^{\gamma}(D^{\sigma}\zeta\ D^{\beta-\sigma}u)\right]\phi$$

$$= (-1)^{|\alpha|}\int \sum_{\sigma\le\beta}\binom{\beta}{\sigma}\left(D^{\gamma+\sigma}\zeta\, D^{\beta-\sigma}u + D^{\sigma}\zeta\cdot D^{\beta+\gamma-\sigma}u\right)\phi$$

$$= (-1)^{|\alpha|} \int \sum_{\sigma \le \beta} \binom{\beta}{\sigma} \Big[\underbrace{D^{\gamma+\sigma} \xi \, D^{\beta-\sigma} u}_{\downarrow} + D^{\sigma} \xi \, D^{\gamma+\beta-\sigma} u \Big] \phi$$

$$\xlongequal[\text{let } \gamma+\sigma=\rho]{\alpha=\beta+\gamma} (-1)^{|\alpha|} \int \left[\sum_{\rho \le \beta+\gamma} \binom{\beta}{\rho-\gamma} D^{\rho} \xi \, D^{\alpha-\rho} u + \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^{\sigma} \xi \, D^{\alpha-\sigma} u \right] \phi$$

$\beta - \sigma = (\beta+\gamma) - (\sigma+\gamma)$
$= \alpha - \rho$

$\sigma = \rho - \gamma$

$$= (-1)^{|\alpha|} \int \left(\sum_{\sigma \le \alpha} \left[\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} \right] D^{\sigma} \xi \, D^{\alpha-\sigma} u \right) \phi$$

$$= (-1)^{|\alpha|} \int \left(\sum_{\sigma \le \alpha} \binom{\beta+\gamma}{\sigma} D^{\sigma} \xi \, D^{\alpha-\sigma} u \right) \phi$$

$$= (-1)^{|\alpha|} \int \sum_{\sigma \le \alpha} \left(\binom{\alpha}{\sigma} D^{\sigma} \xi \, D^{\alpha-\sigma} u \right) \phi$$

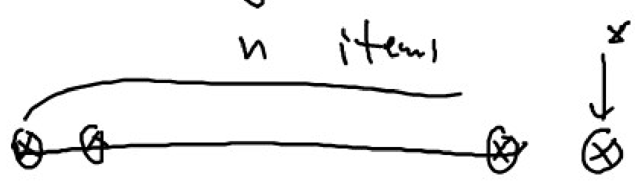
$$\Rightarrow D^{\alpha}(\xi u) = \sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} D^{\sigma} \xi \, D^{\alpha-\sigma} u$$

#.

We have used $\binom{\beta}{\sigma-\gamma} + \binom{\beta}{\sigma} = \binom{\beta+\gamma}{\sigma}$

special case $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

$\binom{n}{k}$ = the # of choice of k items from n items



$\binom{n+1}{k}$ = always pick * + not picking *

$$= \underbrace{\binom{n}{k-1}}_{\substack{\uparrow \\ \text{pick } k-1 \text{ items} \\ \text{from the first} \\ n \text{ items}}} + \underbrace{\binom{n}{k}}_{\substack{\uparrow \\ \text{pick } k \text{ items} \\ \text{from the first} \\ n \text{ items}}}$$