Recall the Sobolev Space

$$W^{k,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{l} \text{the weak derivative } D^\alpha u \text{ exists for } |\alpha| \le k \\ D^\alpha u \in L^p(\Omega) \end{array} \right\}$$

$D^\alpha u$ is the weak partial derivative of u

iff $\displaystyle\int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega (D^\alpha u)\phi$

for $\phi \in C_0^\infty(\Omega)$

In the book, the $W^{k,p}$ norm is defined

by $\displaystyle \|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$

(When $\alpha = 0$, $\|D^0 u\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega)}$)

One can define another norm

$$\|u\|'_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}$$

In fact, these two norms are equivalent

Obviously, $\|u\|_{W^{k,p}(\Omega)} \le \|u\|'_{W^{k,p}(\Omega)}$

Recall that $\displaystyle \left| \sum a_i b_i \right| \le \left(\sum_i a_i^p \right)^{\frac{1}{p}} \cdot \left(\sum_i b_i^q \right)^{\frac{1}{q}}$

when $\frac{1}{p} + \frac{1}{q} = 1$

$$\Rightarrow \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)} \cdot 1 \right) \le \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p} \right)^{\frac{1}{p}} \cdot \underbrace{\left(\sum_{|\alpha| \le k} 1 \right)^{\frac{1}{q}}}_{= C_{k,q}}$$

$$\Rightarrow \|u\|'_{W^{k,p}(\Omega)} \le \|u\|_{W^{k,p}(\Omega)} \cdot \underbrace{C_{k,q}}_{> 0}$$

$\Rightarrow$ $\| \; \|'_{W^{k,p}}$ and $\| \; \|_{W^{k,p}}$ are equivalent

Properties of Weak derivatives.

Th Assume $u, v \in W^{k,p}(\Omega)$

Then (1) For each $\lambda, \mu \in \mathbb{R}$

$\Rightarrow \lambda u + \mu v \in W^{k,p}(\Omega)$

(2) Suppose $|\alpha| + |\beta| \le k$.

Then $D^\alpha u \in W^{k-|\alpha|, p}(\Omega)$,

$D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$

(3) If $V \underset{\text{open}}{\subset} \Omega \Rightarrow W^{k,p}(\Omega) \subset W^{k,p}(V)$

(4) If $\zeta \in C_0^\infty(\Omega)$ then

(a) $\zeta u \in W^{k,p}(\Omega)$

(b) $D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta \cdot D^{\alpha-\beta} u$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$

pf: 1° $u, v \in W^{k,p}(\Omega)$, $\phi \in C_0^\infty(\Omega)$

$\Rightarrow \int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi$, $|\alpha| \le k$

and $\int_\Omega v D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega D^\alpha v \phi$

$\Rightarrow \int_\Omega (\lambda u + \mu v) D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega (\lambda D^\alpha u + \mu D^\alpha v)\phi$ for all $\phi \in C_0^\infty(\Omega)$

$\Rightarrow D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \le k$

Since $D^\alpha u, D^\alpha v \in L^p(\Omega)$ for $|\alpha| \le k$

$\Rightarrow \lambda D^\alpha u + \mu D^\alpha v \in L^p(\Omega)$

$\Rightarrow D^\alpha(\lambda u + \mu v) \in L^p(\Omega)$

$\Rightarrow \lambda u + \mu v \in W^{k,p}(\Omega)$

2° First, Note that

if $\phi \in C_0^\infty(\Omega) \Rightarrow D^\beta \phi \in C_0^\infty(\Omega)$

$|\alpha| + |\beta| \le k$ test ftn

$$\Rightarrow \int u D^{\alpha+\beta}\phi = \int u D^\alpha(\underline{D^\beta \phi}) = (-1)^{|\alpha|} \int D^\alpha u \cdot (D^\beta \phi)$$

$$\| $$

$$(-1)^{|\alpha|+|\beta|} \int D^{(\alpha+\beta)} u \phi$$

$$\Rightarrow \int_\Omega \underset{\underset{w}{\|}}{(D^\alpha u)} \cdot (D^\beta \phi) = (-1)^{|\beta|} \int_\Omega D^{(\alpha+\beta)} u \cdot \phi$$

$$\Rightarrow \int_\Omega w D^\beta \phi = (-1)^{|\beta|} \int_\Omega D^{(\alpha+\beta)} u \cdot \phi \quad \text{for } \phi \in C_0^\infty(\Omega)$$

$\Rightarrow D^\beta w = D^{(\alpha+\beta)} u$

$\Rightarrow D^\beta(D^\alpha u) = D^{(\alpha+\beta)} u$

Similarly $D^\alpha(D^\beta u) = D^{(\alpha+\beta)} u$.

3° If $V \underset{\text{open}}{\subset} \Omega$ and $u \in W^{k,p}(\Omega)$

$\int_V u D^\alpha \phi$ where $\phi \in C_0^\infty(V) \subset C_0^\infty(\Omega)$

$= \int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi = (-1)^{|\alpha|} \int_V D^\alpha u \phi$

$\Rightarrow u \in W^{k,p}(V)$ ($L^p(\Omega) \subset L^p(V)$)