Ch 5 (Evans)
Sobolev Spaces

(Integration by parts formula)

Let $u, \phi \in C^1(\bar{\Omega})$

Then $\displaystyle\int_\Omega \frac{\partial u}{\partial x_i} \phi \, dx = -\int_\Omega u \frac{\partial \phi}{\partial x_i} dx + \int_{\partial\Omega} u \phi \nu^i dS$

where $\nu = (\nu^1, \nu^2, \ldots, \nu^n)$ is the outward unit normal vector on $\partial\Omega$

pf. $\displaystyle\int_\Omega \operatorname{div}\big((0, 0, \ldots, \underset{i\text{-th component}}{u\phi}, 0 \ldots 0)\big)$

$\displaystyle = \int_{\partial\Omega} \underbrace{(0, \ldots 0, u\phi, 0 \ldots 0) \cdot (\nu_1, \ldots, \nu_i, 0 \ldots 0)}_{= u\phi\nu_i} dS(x)$

$\operatorname{div}(0, \ldots 0, u\phi, 0 \ldots 0) = \frac{\partial}{\partial x_i}(u\phi)$

$= \frac{\partial u}{\partial x_i}\phi + u\frac{\partial \phi}{\partial x_i}$

$\displaystyle\Rightarrow \int_\Omega \left(\frac{\partial u}{\partial x_i}\phi + u\frac{\partial \phi}{\partial x_i}\right) = \int_{\partial\Omega} u\phi\nu_i \, dS(x)$

$\displaystyle\Rightarrow \int_\Omega \frac{\partial u}{\partial x_i}\phi = -\int_\Omega u\frac{\partial \phi}{\partial x_i} + \int_{\partial\Omega} u\phi\nu_i \, dS(x)$

Cor:[@] $\displaystyle\int_\Omega \frac{\partial u}{\partial x_i}\phi = -\int_\Omega u\frac{\partial \phi}{\partial x_i}$ if $\phi \in C_0^\infty(\Omega)$

pf: b/c $\phi = 0$ on $\partial\Omega$

Cor: $\displaystyle\int_\Omega D^\alpha u \, \phi = (-1)^{|\alpha|}\int_\Omega u D^\alpha \phi$ — (*)

if $u \in C^k(\bar{\Omega})$, $|\alpha| \le k$

$\phi \in C_0^\infty(\Omega)$

Recall:[@] $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$

$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$

$D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$

@ $C_0^\infty(\Omega) = \{\phi \text{ is smooth in } \Omega \text{ with compact support}\}$

RHS of * makes sense as long as $u D^\alpha \phi$ is integrable.

This leads to the following definition

Def. Suppose $u, v \in L^1_{loc}(\Omega)$ and α is a multi-index, we say v is the α-th weak partial derivative of u, written $D^\alpha u = v$

if $\displaystyle\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|}\int_\Omega v \phi \, dx$

for all test ftns $\phi \in C_0^\infty(\Omega)$.

Lemma: A weak α-th partial derivative of u (if it exists) is uniquely defined up to a set of measure zero.

pf: Suppose $v, \bar{v} \in L^1_{loc}(\Omega)$ are both α-th weak partial derivative of u

$$\Rightarrow \int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega v \phi = (-1)^{|\alpha|} \int_\Omega \bar{v} \phi$$

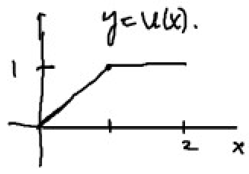
$$\Rightarrow \int_\Omega (v - \bar{v}) \phi = 0 \quad \text{for all } \phi \in C_0^\infty(\Omega)$$

$$\Rightarrow v - \bar{v} = 0 \text{ a.e.}$$

Remark: The weak partial derivative of a fth is unique (in $L^1_{loc}(\Omega)$).

Ex. Let $n=1$, $\Omega = (0,2)$.

$$u(x) = \begin{cases} x & 0 < x \le 1 \\ 1 & 1 \le x < 2 \end{cases}$$



$u \notin C^1(\Omega)$

Define $v = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & 1 < x < 2 \end{cases}$

We want to show that v is the weak 1-st derivative of u

We have to show that

$$\int_0^2 u \phi' \, dx = (-1) \int_0^2 v \phi \, dx \quad \text{for } \phi \in C_0^\infty(\Omega)$$

$$\int_0^2 u \phi' \, dx = \int_0^1 x \phi' dx + \int_1^2 \phi' dx$$

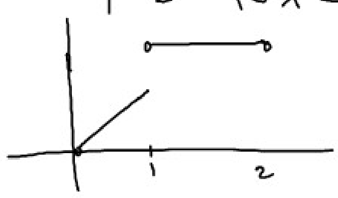
$$= x\phi\Big]_0^1 - \int_0^1 \phi \, dx + \phi(2) - \phi(1)$$

$$= \underbrace{\phi(1)} - 0 - \int_0^1 \phi \, dx + \phi(2) - \underbrace{\phi(1)}$$

$$= - \int_0^1 \phi \, dx \quad \left(\text{b/c } \phi \in C_0^\infty((0,2)) \Rightarrow \phi(2) = 0 \right)$$

$$= - \int_0^2 v \phi \, dx \quad \left(\text{b/c } v = \begin{cases} 1 & 0 < x \le 1 \\ 0 & 1 < x < 2. \end{cases} \right)$$

Ex: $u(x) = \begin{cases} x & 0 < x \le 1 \\ 2 & 1 < x < 2 \end{cases}$



The weak derivative of u doesn't exist. (see example 2 on p244).

Definition of Sobolev Spaces.

① $W^k(\Omega) = \left\{ u \in L^1_{loc}(\Omega) \;\middle|\; \begin{array}{l} \text{the weak derivative} \\ D^\alpha u \text{ exists for} \\ |\alpha| \le k \end{array} \right\}$

Remark. $C^k(\Omega) \subset W^k(\Omega)$

② $W^{k,p}(\Omega) = \left\{ u \in W^k(\Omega) \;\middle|\; \begin{array}{l} D^\alpha u \in L^p(\Omega) \\ \text{for all } |\alpha| \le k \end{array} \right\}$

($|\alpha| = 0$ means $u \in L^p(\Omega)$)

$W^{k,p}(\Omega) \subset L^p(\Omega)$

Def. If $u \in W^{k,p}(\Omega)$, we define the norm

$$\|u\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}$$

$$\left(= \sum_{|\alpha| \le k} \left(\int |D^\alpha u|^p dx \right)^{\frac{1}{p}} \right.$$

$1 \le p < \infty$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha| \le k} \operatorname{ess\,sup}_{\Omega} |D^\alpha u|$$

Remark. We'll show that

$\left(W^{k,p}, \| \; \|_{W^{k,p}(\Omega)} \right)$

is a Banach space later

Def. ① Let $\{u_m\}_{m=1}^{\infty}$, $u \in W^{k,p}(\Omega)$

We say u_m converges to u in $W^{k,p}(\Omega)$, write $u_m \longrightarrow u$ in $W^{k,p}(\Omega)$

if $\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$

② $u_m \longrightarrow u$ in $W^{k,p}_{loc}(\Omega)$

if $u_m \to u$ in $W^{k,p}(V)$ for each $V \subset\subset \Omega$.

Def. $W_0^{k,p}(\Omega) =$ the closure of $C_0^\infty(\Omega)$ in $\left(W^{k,p}(\Omega), \| \; \|_{W^{k,p}} \right)$.

$u \in W_0^{k,p}(\Omega) \Leftrightarrow \exists\, u_m \in C_0^\infty(\Omega)$

st $\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(\Omega)} = 0$

$H_0^k(\Omega) = W_0^{k,2}(\Omega) \subset L^2(\Omega)$

We'll show that this is a Hilbert space later.