HW $\cancel{2}$4(a) should be

$$\Delta(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4)$$

$$= 2\phi^4 |\mathrm{Hess}(v)|^2 + 4\sum_{i,j} \frac{\partial \phi^4}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j}$$

$$+ w\Delta\phi^4 + 2w\nabla v \cdot \nabla\phi^4$$

(4c) $\displaystyle \sup_{B_r} \frac{|\nabla u|}{u} \le \frac{C}{r}$

Last time, $\delta = \min\left(\frac{\delta_1}{2}, \delta_2\right)$

$\overline{\overline{\neq}}$ not δ_1

Dirichlet Principle:

"Least action principle":
The eqs in "physics" or "Math",
should be the minimizer of
certain energy (functional).

Consider the energy functional

$$I(w) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - wf\right) dx$$

where w belongs to $(f \in C^0(\bar\Omega))$
the admissable set

$$A = \{ w \in C^2(\bar\Omega) \mid w = g \text{ on } \partial\Omega \}$$

Th: Assume $u \in C^2(\bar\Omega)$
and
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases} \quad (**)$$

Then $I(u) = \min_{w \in A} I(w) \leftarrow (*)$

Conversely if $u \in A$ satisfies $*$
then u solves the BVP $**$.

Remark: u is a minimizer of the functional I in A
$\Leftrightarrow$ u solves BVP in A.

pf: Prove $I(u) = \min_{w \in A} I(w)$

$$\int_\Omega \underset{\overset{\|}{0}}{(\Delta u + f)}(u - w) = 0$$

$$\underbrace{\int (\Delta u)(u-w)}_{\|} + \int f(u-w) = 0$$

$$\int \operatorname{div}\left(\nabla u\,(u-w)\right) - \nabla u \cdot (\nabla u - \nabla w)$$

By assumpt $u = w = g$ on $\partial\Omega$
$\Rightarrow$ $u - w = 0$ on $\partial\Omega$
$\Rightarrow$ $\int_\Omega \operatorname{div}(\nabla u (u-w)) = 0$

$$\int -|\nabla u|^2 + \nabla u \cdot \nabla w + \int f(u-w) = 0$$

$$\Rightarrow \int |\nabla u|^2 - fu + fw = \int \nabla u \cdot \nabla w$$

$$\leq \int \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2$$

$$\Rightarrow \int \left(\frac{1}{2}|\nabla u|^2 - fu\right) \leq \int \left(\frac{1}{2}|\nabla w|^2 - fw\right)$$

$\Rightarrow$ $I(u) \leq I(w)$ for all $w \in A$

$\Rightarrow$ $I(u) = \min_{w \in A} I(w)$

Now we want to prove that if $I(u) = \min_{w \in A} I(w)$

$\Rightarrow$ u solves $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

Fix $\phi \in C_0^\infty(\Omega)$ $\left(\Rightarrow \phi = 0 \text{ near } \partial\Omega \right)$

Consid $u + \tau\phi$ where $\tau \in \mathbb{R}$

$\Rightarrow$ $u + \tau\phi|_{\partial\Omega} = u|_{\partial\Omega} = g$

Also $u + \tau\phi \in C^2(\bar{\Omega})$

$\Rightarrow$ $u + \tau\phi \in A$

Let $F(\tau) = I(u + \tau\phi)$, $\tau \in \mathbb{R}$

$\Rightarrow$ $F(0) = I(u)$ and achives its minimum at $\tau = 0$

Also F is smooth $\tau \Rightarrow F'(0) = 0$

Recall $I(w) = \int \frac{1}{2}|\nabla w|^2 - wf$

$\Rightarrow$ $F(\tau) = I(u + \tau\phi)$

$= \int \frac{1}{2} |\nabla u + \tau \nabla\phi|^2 - (u + \tau\phi) f$

$= \int \frac{1}{2} \left(|\nabla u|^2 + \tau^2 |\nabla\phi|^2 + 2\tau \nabla u \cdot \nabla\phi \right)$

$- \int (u + \tau\phi) f$ indep of τ

$\Rightarrow$ $F'(\tau) = \int \tau|\nabla\phi|^2 + \nabla u \cdot \nabla\phi - \phi f$

$F'(0) = \int_\Omega \nabla u \cdot \nabla\phi - \phi f$

$F'(0) = 0 \Leftrightarrow \int_\Omega \nabla u \cdot \nabla\phi - \phi f = 0$

$\Rightarrow$ $\int_\Omega \underbrace{\operatorname{div}(\phi \nabla u)}_{=0} - \phi\Delta u - \phi f = 0$

b/c $\left(\begin{array}{l} \phi \in C_0^\infty(\Omega) \\ \phi = 0 \text{ on } \partial\Omega \end{array} \right)$

$\Rightarrow$ $\int_\Omega \phi \underbrace{(\Delta u + f)}_{\text{cts fth}} = 0$ for any $\phi \in C_0^\infty(\Omega)$

$\Rightarrow$ $\Delta u + f = 0$ in Ω

$\Rightarrow$ $-\Delta u = f$

b/c $u \in A \Rightarrow u|_{\partial\Omega} = g$

$\Rightarrow$ u solves BVP.

Remark: u solves BVP

$\Rightarrow$ $\underbrace{\int_\Omega \nabla u \cdot \nabla\phi}_{\text{defined when } u \in C^1(\Omega)} = \int_\Omega \phi f$

$\langle u, \phi \rangle = \int_\Omega \nabla u \cdot \nabla\phi$ is a inner product

$\phi \longmapsto \int \phi f$ a linear functional

Consider the completion of $C_0^\infty(\Omega)$ wrt this inner product

Sobolev Sp