Recall that we derive the formula

(*)
$$u(x) = \frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\,dS(y)$$

by assuming that we have a sol to the boundary value problem

$$\begin{cases} \Delta u = 0 \text{ in } B(0,r) \\ u|_{\partial B(0,r)} = g \end{cases}$$

We'll show that the formula indeed solves the BVP.

Remark ① Recall $G(x,y) = \Phi(y-x) - \phi^x(y)$

where $\begin{cases} \Delta_y \phi^x(y) = 0 \quad \text{in } \Omega \\ \phi^x(y) = \Phi(y-x) \text{ when } y \in \partial\Omega \end{cases}$

$\Rightarrow \Delta_y G(x,y) = 0$

Since $G(x,y) = G(y,x)$

$\Rightarrow \Delta_x G(x,y) = 0$

$G(x,y)$ is harmonic in x and y. when $x \neq y$

② The sol of $\begin{cases} \Delta u = 0 \text{ in } B(0,r) \\ u|_{\partial B(0,r)} = 1 \end{cases}$

is $u = 1$

$\Rightarrow$ By Repsentation formula *

$$\Rightarrow 1 = \frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)} \frac{1}{|x-y|^n}\,dS(y)$$

for $x \in B(0,r)$.

Thm: Assume $g \in C^0(\partial B(0,r))$ and define u by

$$u(x) = \frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)} \frac{g(y)\,dS(y)}{|x-y|^n}.$$

for $x \in B(0,r)$

Then (1) $u \in C^\infty(B(0,r))$

(2) $\Delta u = 0$ in $B(0,r)$

(3) $\lim_{x\to x_0} u(x) = g(x_0)$

where $x_0 \in \partial B(0,r)$

pf: 1° b/c $\underbrace{\frac{r^2-|x|^2}{n\alpha_n r}}_{\text{smooth in } x}\underbrace{\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\,dS(y)}_{\text{smooth in } x \text{ when } x\in B(0,r)}$

$$\left(\frac{\partial}{\partial x_i}\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\,dS(y) = \int_{\partial B(0,r)} \frac{\partial}{\partial x_i}\left(\frac{g(y)}{|x-y|^n}\right)dS(y)\right)$$

$\Rightarrow$ u is smooth in $B(0,r)$

2° Recall $u(x) = -\int_{\partial B(0,r)} g(y)\frac{\partial G(x,y)}{\partial \nu}\,dS(y)$

$$= -\int g(y)\left(\sum_{i=1}^n \frac{\partial G(x,y)}{\partial y_i}\cdot\frac{y_i}{r}\right)dS(y)$$

$$\left(b/c\ \frac{\partial G}{\partial \nu} = \nabla_y G(x,y)\cdot \underset{(\shortparallel)}{\nu} = \nabla_y G(x,y)\cdot\frac{y}{r}\right.$$

$$\Delta_x u(x) = -\int g(y)\left(\sum_{i=1}^n \frac{\partial}{\partial y_i}\Big(\underset{\shortparallel\atop 0}{\Delta_x G(x,y)}\cdot\frac{y_i}{r}\right)dS(y)$$

$= 0$ when $x \in B(0,r)$

3° Recall that

$$1 = \frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)} \frac{1}{|x-y|^n}\,dS(y)$$

(This implies that if $x \to \partial B(0,r)$

$\Rightarrow r^2-|x|^2 \to 0$

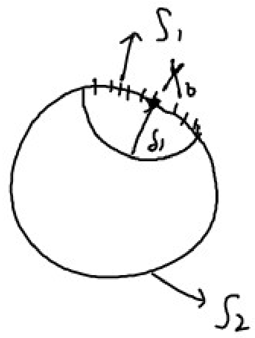
$\Rightarrow \int_{\partial B(0,r)} \frac{1}{|x-y|^n}\,dS(y) \to \infty$)

Fix $x_0 \in \partial B(0,r)$.

$$\Rightarrow g(x_0) = \frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)} \frac{g(x_0)}{|x-y|^n}\,dS(y)$$

$$\Rightarrow u(x) - g(x_0) = \frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)} \frac{g(y)-g(x_0)}{|x-y|^n}\,dS(y)$$

$$u(x)-g(x_0)=\frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)}\frac{g(y)-g(x_0)}{|x-y|^n}\,dS(y)$$



b/c g is cts

We can find $\delta_1>0$

s.t if $|y-x_0|<\delta_1$

$$\Longrightarrow \quad |g(y)-g(x_0)|<\frac{\varepsilon}{2}$$

Let $S_1=\{y \mid y\in\partial B(0,r),\ |y-x_0|<\delta_1\}$

$S_2=\{y \mid y\in\partial B(0,r),\ |y-x_0|\geq\delta_1\}$.

$$\Rightarrow |u(x)-g(x_0)|=\frac{r^2-|x|^2}{n\alpha_n r}\left|\int_{S_1}\frac{g(y)-g(x_0)}{|x-y|^n}+\int_{S_2}\frac{g(y)-g(x_0)}{|x-y|^n}\right|$$

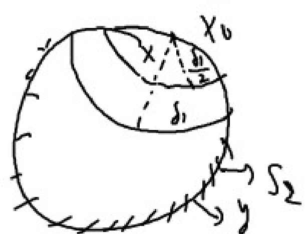
$$\leq\frac{\varepsilon}{2}\left(\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_1}\frac{1}{|x-y|^n}\right)+\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_2}\frac{|g(y)-g(x_0)|}{|x-y|^n}$$

$$\leq\frac{\varepsilon}{2}\left(\underbrace{\frac{r^2-|x|^2}{n\alpha_n r}\int_{\partial B(0,r)}\frac{1}{|x-y|^n}}_{=1}\right)+$$

$$\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_2}\frac{|g(y)-g(x_0)|}{|x-y|^n}$$

$$\leq\frac{\varepsilon}{2}+2M\cdot\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_2}\frac{1}{|x-y|^n}$$

(g is cts $\Rightarrow$ $|g(y)|\leq M$, $y\in\partial B(0,r)$)



Suppose $\quad |x_0-x|<\frac{\delta_1}{2}\Longrightarrow |x-y|\geq\frac{\delta_1}{2}$

$$\Longrightarrow\quad\frac{1}{|x-y|^n}\leq\left(\frac{2}{\delta_1}\right)^n$$

$$\Longrightarrow\quad 2M\cdot\frac{r^2-|x|^2}{n\alpha_n r}\int_{S_2}\frac{1}{|x-y|^n}$$

$$\leq 2M\cdot\left(\frac{2}{\delta_1}\right)^n\cdot\left(n\alpha_n r^{n-1}\right)\cdot\frac{r^2-|x|^2}{n\alpha_n r}$$

if $\quad |x-x_0|<\frac{\delta_1}{2}$

Chose δ_2 s.t if $|x-x_0|<\delta_2$ ($|x_0|=r$)

$$\Rightarrow\left(2M\cdot\left(\frac{2}{\delta_1}\right)^n\left(n\alpha_n r^{n-1}\right)\frac{r^2-|x|^2}{n\alpha_n r}\right)<\frac{\varepsilon}{2}$$

Choose $\delta=\min(\delta_2,\delta_1)$

$\Rightarrow \quad |x-x_0|<\delta\Rightarrow |u(x)-g(x_0)|<\varepsilon$

$\Rightarrow \quad \lim_{x\to x_0}u(x)=g(x_0)$

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