Recall that:

Last week, we derived the Green's ftn on the unit ball

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$$

where $\tilde{x} = \frac{x}{|x|^2}$

and $\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| \\ \frac{1}{n(n-2)\alpha_n} |x|^{2-n} \end{cases}$

Now we can use the formula of Green's ftn to solve the boundary problem

$$* \begin{cases} \Delta u = 0 & \text{in } B(0,1) \\ u = g & \text{in } \partial B(0,1) \end{cases}$$

Thm: The sol for $*$ is

$$u(x) = \frac{1-|x|^2}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n} \, dS(y)$$

pf: Recall from the representation formula using Green's ftn for the sol of $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g & \text{in } \partial\Omega \end{cases}$

$$\Rightarrow u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G(x,y)}{\partial \nu} \, dS(y) + \int_{\Omega} f(y) G(x,y) \, dy$$

Now $\Delta u = 0$ in $B(0,1)$, $u|_{\partial B(0,1)} = g$.

and $G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x}))$

$$\Rightarrow u(x) = -\int_{\partial B(0,1)} g(y) \underbrace{\frac{\partial G(x,y)}{\partial \nu}} \, dS(y)$$

Claim: $\left(\frac{\partial G(x,y)}{\partial \nu} = \nabla_y G(x,y) \cdot \nu \right)$

$$\nabla_y G(x,y) = \frac{(|x|^2-1)}{n\alpha_n |x-y|^n} y$$

$\nu = y$ when $y \in \partial B(0,1)$

In $B(0,1)$, $\nu = y$ on $\partial B(0,1)$

$$\Rightarrow \frac{\partial G(x,y)}{\partial \nu} = \nabla_y G(x,y) \cdot \nu$$

$$= \frac{(|x|^2-1)}{n\alpha_n |x-y|^n} y \cdot y$$

$$= \frac{|x|^2-1}{n\alpha_n |x-y|^n} \quad \text{when } y \in \partial B(0,1) \ (|y|^2 = 1)$$

$$\Rightarrow u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G(x,y)}{\partial \nu} \, dS(y)$$

$$= -\int_{\partial B(0,1)} g(y) \left(\frac{|x|^2-1}{n\alpha_n |x-y|^n} \right) dS(y)$$

$$= \frac{1-|x|^2}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n} \, dS(y)$$

pf of the claim

$$\nabla_y G(x,y) = \frac{(|x|^2-1)}{n\alpha_n |x-y|^n} y \quad \text{when } |y|=1$$
$$(y \in \partial B(0,1))$$

Recall $G(x,y) = \underline{\Phi(y-x)} - \underline{\Phi(|x|(y-\tilde{x}))}$

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| & n=2 \\ \frac{1}{n(n-2)\alpha_n} |x|^{2-n} & n \geq 3 \end{cases}$$

In the following, we assume $(n \geq 3)$

$$\frac{\partial \Phi(y-x)}{\partial y_i} = \frac{1}{n(n-2)\alpha_n}(2-n)|y-x|^{1-n} \cdot \frac{\partial}{\partial y_i}|y-x|$$
$$= \frac{-1}{n\alpha_n}|y-x|^{1-n} \cdot \frac{y_i-x_i}{|y-x|}$$
$$= \frac{x_i-y_i}{n\alpha_n |y-x|^n}$$

$(f(cy))' = c f'(cy)$

$$\frac{\partial}{\partial y_i}\Phi(\underbrace{|x|(y-\tilde{x})}_{z}) = |x| \left.\frac{\partial \Phi}{\partial z_i}\right|_{|x|(y-\tilde{x})}$$
$$= |x| \frac{|x|(\tilde{x}_i - y_i)}{n\alpha_n \, ||x|(y-\tilde{x})|^n}$$

Recall that $\tilde{x} = \frac{x}{|x|^2}$

and $\left||x|(y-\tilde{x})\right| = |y-x|$ when $|y|=1$

$$\Rightarrow \frac{\partial}{\partial y_i}\Phi(|x|(y-\tilde{x})) = \frac{|x|^2\left(\frac{x_i}{|x|^2} - y_i\right)}{n\alpha_n |x-y|^n}$$
$$= \frac{(x_i - |x|^2 y_i)}{n\alpha_n |x-y|^n} \quad \text{when } |y|=1$$

$$\Rightarrow \frac{\partial G(x,y)}{\partial y_i} = \frac{\partial \Phi(y-x)}{\partial y_i} - \frac{\partial}{\partial y_i}\Phi(|x|(y-\tilde{x}))$$
$$= \frac{x_i - y_i - (x_i - |x|^2 y_i)}{n\alpha_n |x-y|^n}$$
$$= \frac{(|x|^2-1)y_i}{n\alpha_n |x-y|^n}$$

$$\Rightarrow \nabla_y G(x,y) = \frac{(|x|^2-1)y}{n\alpha_n |x-y|^n} \quad \text{when } |y|=1$$

#.

Thm: $u(x) = \dfrac{r^2 - |x|^2}{n\alpha_n r} \displaystyle\int_{\partial B(0,r)} \dfrac{g(y)}{|x-y|^n}\, dS(y)$

where u solves
$$\begin{cases} \Delta u = 0 & \text{in } B(0,r) \\ u|_{\partial B(0,r)} = g \end{cases}$$

pf: Let $w(x) := u(rx)$.

Then $\begin{cases} \Delta w = 0 & \text{in } B(0,1) \\ w(y) = g(ry) & \text{when } y \in \partial B(0,1) \end{cases}$

By previous Th,

$$\Rightarrow \underset{u(rx)}{\overset{\|}{w(x)}} = \frac{1-|x|^2}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(ry)}{|x-y|^n}\, dS(y)$$

Let $rx = z \Rightarrow x = \frac{z}{r}$

$$\Rightarrow u(z) = \frac{1 - \frac{|z|^2}{r^2}}{n\alpha_n} \int_{\partial B(0,1)} \frac{g(ry)}{\left|\frac{z}{r} - y\right|^n}\, dS(y)$$

$$= \frac{r^2 - |z|^2}{n\alpha_n r^2} \int_{\partial B(0,1)} \frac{g(ry)\, r^n}{|z - ry|^n}\, dS(y)$$

Let $\bar{y} = ry \Rightarrow dS(\bar{y}) = r^{n-1} dS(y)$

$$\Rightarrow u(z) = \frac{r^2 - |z|^2}{n\alpha_n} \cdot \frac{r^n}{r^2 \cdot r^{n-1}} \int_{\partial B(0,r)} \frac{g(\bar{y})}{|z - \bar{y}|^n}\, dS(\bar{y})$$

$$= \frac{r^2 - |z|^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(\bar{y})}{|z - \bar{y}|^n}\, dS(\bar{y})$$

#.

Def: The ftn $K(x,y) = \dfrac{r^2 - |x|^2}{n\alpha_n r\, |x-y|^n}$

is called the Poisson Kernel

for the ball $B(0,r)$.

$(\ x \in B(0,r),\ y \in \partial B(0,r))$

Remark: If $x = 0$

$$\Rightarrow u(0) = \frac{r^2}{n\alpha_n r} \int_{\partial B(0,r)} \frac{g(y)}{\underbrace{|y|^n}_{= r^n}}\, dS(y)$$

$$= \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} g(y)\, dS(y)$$

$(\ u = g$ on $\partial B(0,r)$).

$$= \frac{1}{n\alpha_n r^{n-1}} \int_{\partial B(0,r)} u(y)\, dS(y)$$

$\Rightarrow$ We get the mean-value equality

② In the formula above,

We assume $u \in C^2(\overline{B(0,r)})$.