In the following, we'll construct Green's fn for the unit ball.

Def. $x \in \mathbb{R}^n \setminus \{0\}$, the point $\tilde{x} = \frac{x}{|x|^2}$ is called the point dual to x with respect to $\partial B(0,1)$.

The mapping $x \longmapsto \tilde{x}$ is inversion thru the unit sphere.

Remark: ① If $\|x\| = R \Rightarrow \|\tilde{x}\| = \frac{1}{R}$

② If $|x| < 1 \Rightarrow |\tilde{x}| > 1$

To construct Green's fn on unit ball we have to find a corrector $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta_y \phi^x(y) = 0 & \text{in } B(0,1) \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial B(0,1) \end{cases}$$

Lemma: $\|y - \tilde{x}\| = \frac{\|x-y\|}{\|x\|}$ when $y \in \partial B(0,1)$ (or $\|y\|=1$)

pf: $\|y-\tilde{x}\|^2 = \|y - \frac{x}{\|x\|^2}\|^2$

$= \|y\|^2 - 2\frac{y\cdot x}{\|x\|^2} + \frac{\|x\|^2}{\|x\|^4}$

$\underset{\|y\|=1}{=} 1 - 2\frac{y\cdot x}{\|x\|^2} + \frac{1}{\|x\|^2}$

$= \frac{1}{\|x\|^2}\left(\underset{\|y\|}{1} - 2y\cdot x + \|x\|^2\right)$

$= \frac{1}{\|x\|^2}\left(\|y\|^2 - 2y\cdot x + \|x\|^2\right)$

$= \frac{1}{\|x\|^2}\|y-x\|^2$

$\Rightarrow \|y-\tilde{x}\| = \frac{\|y-x\|}{\|x\|}$.

Remark:

② If one considers $\|y - ax\|$ so that $\frac{\|y-ax\|}{\|y-x\|}$ is indep of y, then $a = \frac{1}{\|x\|^2}$

①

$$\frac{\|y-\tilde{x}\|}{\|y-x\|} = \frac{1}{\|x\|} \leftarrow \text{indep of } y \text{ for } |y|=1$$

$\|x\|\,\|y-\tilde{x}\| = \|y-x\|$

if $x \in B(0,1)$ and $x \neq 0 \Rightarrow \tilde{x} \notin B(0,1)$

($\underset{\text{In fact}}{\tilde{x} \in \mathbb{R}^n \setminus \overline{B(0,1)}}$)

Define $\phi^x(y) = \Phi(\|x\|(y-\tilde{x}))$

Recall that if u is harmon $\Rightarrow u(cx)$ is also harmonic

$$\left(\Delta u(cx) = c^2 \Delta_y u(\underset{y}{cx})\right)$$

So $\phi^x(y)$ is harmonic ($\because \Phi$ is harmonic)

$\Rightarrow \Delta_y \phi^x(y) = 0 \quad \left(\Delta_y\left(\Phi(\|x\|(y-\tilde{x}))\right) = \|x\|^2 \Delta_z \Phi(z) = 0\right)$

Also $\phi^x(y)$

$= \Phi(\|x\|(y-\tilde{x})) = \Phi(\|x\|\,\|y-\tilde{x}\|)$

$= \Phi(\|y-x\|) = \Phi(y-x)$ when $y \in \partial B(0,1)$

So $\phi^x(y) = \Phi\left(\|x\|(y-\bar{x})\right)$ is smooth.

satisfies

$$\begin{cases} \Delta_y \phi^x(y) = 0 & \text{in } B(0,1) \\ \phi^x(y) = \Phi(y-x) & \text{when } y \in \partial B(0,1) \end{cases}$$

(Recall that $\Phi\left(\|x\|(y-\bar{x})\right)$ is not defined at $\bar{x} \notin B(0,1)$)

Def: Green's ftn for the unit ball is

$$G(x,y) = \Phi(y-x) - \Phi\left(\|x\|(y-\bar{x})\right)$$
$$x, y \in B(0,1)$$

Remark: The def only works for $x \neq 0$

But if $x = 0$ $\Phi(y-x) = \Phi(y)$

$= \text{constant} = C$

when $y \in \partial B(0,1)$

We can choose $\phi^x(y) \equiv \text{constant}$

when $x = 0$.