Recall Green's representation formula

$$u(x) = \int_{\partial\Omega} \left(\Phi(y-x)\frac{\partial u(y)}{\partial \nu} - u(y)\frac{\partial \Phi(y-x)}{\partial \nu} \right) dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\, dy$$

Suppose u solves $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

By Green's representation formula,

$$\Rightarrow u(x) = \int_{\partial\Omega} \Big(\Phi(y-x)\underbrace{\frac{\partial u(y)}{\partial \nu}}_{\Downarrow} - g(y)\frac{\partial \Phi(y-x)}{\partial \nu} \Big) dS(y) + \int_{\Omega} \Phi(y-x) f(y)\, dy$$

This is unknown.

Need to use the value of u in a nbd of $\partial\Omega$ to compute $\frac{\partial u}{\partial \nu}$. But this info is not given.

So we need to find some way to get rid of $\int_{\partial\Omega} \Phi(y-x)\frac{\partial u(y)}{\partial \nu}$.

Green's ftn:

For fixed $x \in \Omega$, define a corrector ftn $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta_y \phi^x(y) = 0 \\ \phi^x(y) = \Phi(y-x) \text{ when } y \in \partial\Omega \end{cases}$$

Note that $\phi^x(y)|_{\partial\Omega}$ is a "smooth ftn".

The solution $\phi^x(y)$ is smooth.

Apply Green's 2nd Identity

$$\int_{\Omega} \Big[u(y)\underbrace{\Delta \phi^x(y)}_{=0} - \phi^x(y)\Delta u(y) \Big] dy$$

$$= \int_{\partial\Omega} \Big[u(y)\frac{\partial \phi^x(y)}{\partial \nu} - \underbrace{\phi^x(y)}_{=\Phi(y-x)}\frac{\partial u(y)}{\partial \nu} \Big] dS(y)$$

$$\Rightarrow -\int \phi^x(y)\Delta u(y)\, dy$$

$$① \quad = \int_{\partial\Omega} \Big[u(y)\frac{\partial \phi^x(y)}{\partial \nu} - \underbrace{\Phi(y-x)\frac{\partial u(y)}{\partial \nu}}_{} \Big] dS(y)$$

Recall

$$② \quad u(x) = \int_{\partial\Omega} \Big[\underbrace{\Phi(y-x)\frac{\partial u(y)}{\partial \nu}}_{} - u(y)\frac{\partial \Phi(y-x)}{\partial \nu} \Big] dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y)\, dy$$

$$①+② \Rightarrow u(x) = \int_{\partial\Omega} u(y)\left(\frac{\partial}{\partial \nu}\big(\phi^x(y) - \Phi(y-x)\big) \right) dS(y) + \int_{\Omega} \big(\phi^x(y) - \Phi(y-x)\big)\Delta u(y)\, dy$$

Def. Green's ftn for Ω is

$$G(x,y) = \Phi(y-x) - \underset{\uparrow}{\phi^x(y)}$$

defined earlier.

$$\Rightarrow u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)\, dS(y) - \int_{\Omega} G(x,y)\Delta u(y)\, dy$$

Remark.

① If $\begin{cases} \Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

$$\Rightarrow u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu}\, dS(y) + \int_{\Omega} G(x,y) f(y)\, dy$$

② If u solves $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

$$\Rightarrow u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G(x,y)}{\partial \nu}\, dS(y)$$

Th (Symmetry of Green's ftn).

For $x, y \in \Omega$, $x \neq y$,

we have $G(x,y) = G(y,x)$

pf: Fix $x, y \in \Omega$, $x \neq y$.

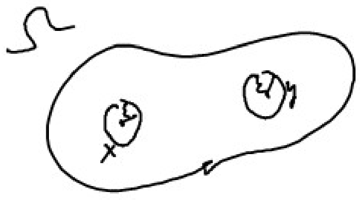
Write $V(z) = G(x,z)$ and $W(z) = G(y,z)$.

Recall that $G(x,z) = \underbrace{\Phi(z-x)}_{\text{smooth in } \Omega \setminus \{x\}} - \underbrace{\phi^x(z)}_{\text{smooth}}$

$\Rightarrow$ V is smooth in $\Omega \setminus \{x\}$.

W is smooth in $\Omega \setminus \{y\}$.

$\Rightarrow$ V, W is smooth in $\Omega \setminus \{x, y\}$



Let ε small enough so that

$B(x,\varepsilon) \cap B(y,\varepsilon) = \phi$,

$B(x,\varepsilon) \subset \Omega$, $B(y,\varepsilon) \subset \Omega$

$\Rightarrow$ V and W are smooth in $\Omega \setminus (B(x,\varepsilon) \cup B(y,\varepsilon))$

$\Rightarrow$ Apply Green's 2nd identity in

$$0 = \int_{\Omega \setminus (B(x,\varepsilon) \cup B(y,\varepsilon))} \Big(V(z) \underbrace{\Delta_z W}_{=0} - W(z) \underbrace{\Delta_z V}_{=0}\Big)\, dz$$

$$= \int_{\partial(\Omega \setminus B(x,\varepsilon) \cup B(y,\varepsilon))} \left(V \frac{\partial W}{\partial \nu} - W \frac{\partial V}{\partial \nu}\right) dS(z)$$

(From def, $\Delta_z V = 0$ in $\Omega \setminus \{x\}$

$\Delta_z W = 0$ in $\Omega \setminus \{y\}$.

$V(z) = \Phi(x,z) - \phi^x(z) = 0$ on $\partial\Omega$

$W(z) = \Phi(y,z) - \phi^y(z) = 0$ on $\partial\Omega$

$$\Rightarrow \int_{\partial B(x,\varepsilon)} \left(V \frac{\partial W}{\partial \nu} - W \frac{\partial V}{\partial \nu}\right) + \int_{\partial B(y,\varepsilon)} \left(V \frac{\partial W}{\partial \nu} - W \frac{\partial V}{\partial \nu}\right) = 0$$

$$\Rightarrow \int_{\partial B(x,\varepsilon)} \left(W \frac{\partial V}{\partial \nu} - V \frac{\partial W}{\partial \nu}\right) dS(y)$$

$$= \int_{\partial B(y,\varepsilon)} \left(V \frac{\partial W}{\partial \nu} - W \frac{\partial V}{\partial \nu}\right) dS(y)$$

Claim: $\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \left(W \frac{\partial V}{\partial \nu} - V \frac{\partial W}{\partial \nu}\right) dS(y)$

$= G(y,x)$ and

$\lim_{\varepsilon \to 0} \int_{\partial B(y,\varepsilon)} \left(V \frac{\partial W}{\partial \nu} - V \frac{\partial V}{\partial \nu}\right) dS(y)$

$= G(x,y)$

$\Longrightarrow$ $G(y,x) = G(x,y)$