PDE

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Plan of today's talk

- A overview of the proof of Green's representation formula
- Green's function
- Properties of Green's function

Green's representation formula

Green's representation formula Let $u \in C^2(\overline{\Omega})$.
Then $u(x) =$
$\int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi(y-x)}{\partial \nu})dS(y) - \int_{\Omega}\Phi(y-x)\triangle u(y)dy$

Key points in the proof:
1. Apply Green's second identity to $\Omega \setminus B(x,\epsilon)$ to get
$\int_{\Omega\setminus B(x,\epsilon)} u(y)\triangle\Phi(y-x) - \Phi(y-x)\triangle u(y)dy =$
$\int_{\partial(\Omega\setminus B(x,\epsilon))}(u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u}{\partial \nu}(y))dS(y)$

2. Use the fact that $\triangle_y\Phi(y-x) = 0$ (regard this as a function of y) to get
$\int_{\Omega\setminus B(x,\epsilon)} -\Phi(y-x)\triangle u(y)dy =$
$\int_{\partial(\Omega\setminus B(x,\epsilon))}(u(y)\frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x)\frac{\partial u}{\partial \nu}(y))dS(y)$

3. Note that $\partial(\Omega \setminus B(x,\epsilon) = \partial\Omega \cup \partial B(x,\epsilon)$. So we have
$\int_{\Omega\setminus B(x,\epsilon)} -\Phi(y-x)\triangle u(y)dy$
$= \int_{\partial\Omega}(u(y)\frac{\partial\Phi(y-x)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y)$
$+ \int_{\partial B(x,\epsilon)}(u(y)\frac{\partial\Phi(y-x)}{\partial\nu} - \Phi(y-x)\frac{\partial u}{\partial\nu}(y))dS(y)$.
Note that the unit normal vector is the exterior normal vector with respect to $\Omega \cup \partial B(x,\epsilon)$.

4. Show that $\int_{\partial B(x,\epsilon)} u(y)\frac{\partial\Phi(y-x)}{\partial\nu}dS(y) = \frac{\int_{\partial B(x,\epsilon)} u(y)dS(y)}{n\alpha_n\epsilon^{n-1}}$ and
$|\int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial\nu}(y)dS(y)| \leq \begin{cases} C\epsilon\ln\varepsilon & n=2 \\ C\epsilon & \text{if } n\geq 3 \end{cases}$
In particular, $\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} u(y)\frac{\partial\Phi(y-x)}{\partial\nu}dS(y) = u(x)$
and $\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial\nu}(y)dS(y) = 0$

5. Now $\lim_{\epsilon \to 0} \int_{\Omega \setminus B(x,\epsilon)} -\Phi(y-x) \triangle u(y) dy$
$= \int_{\partial \Omega} (u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)) dS(y)$
$+ \lim_{\epsilon \to 0} \int_{\partial B(x,\epsilon)} (u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)) dS(y)$ gives

$\int_{\Omega} -\Phi(y-x) \triangle u(y) dy$
$= \int_{\partial \Omega} (u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)) dS(y) + u(x)$ which is equivalent to Green's representation formula.

Recall (Green's representation formula) $u(x) = \int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi(y-x)}{\partial \nu})dS(y) - \int_\Omega \Phi(y-x)\triangle u(y)dy$

Suppose u satisfies $\begin{cases} -\triangle u = f & \text{in } \Omega \\ u|_{\partial\Omega} = g \end{cases}$

By Green's representation formula, we have $u(x) = \int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial \nu}(y) - g(y)\frac{\partial \Phi(y-x)}{\partial \nu})dS(y) + \int_\Omega \Phi(y-x)f(y)dy$.

The drawback of this formula is that it involves the normal derivative of u on the boundary (which is unknown).

So we must find someway to remove $\int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial \nu}(y)dS(y)$.

For fixed $x \in \Omega$, define a corrector function $\phi^x = \phi^x(y)$ a function of y solving $\begin{cases} \triangle_y \phi^x(y) = 0 & \text{in } \Omega \\ \phi^x(y) = \Phi(y-x) & \text{when } y \in \partial\Omega \end{cases}$

Note that $\Phi(y-x)$ is a smooth function in y when $y \in \partial\Omega$ and $x \in \Omega$.

By Green's second identity

$$\int_\Omega u(y)\triangle\phi^x(y) - \phi^x(y)\triangle u(y)dy$$

$$= \int_{\partial\Omega} (u(y)\frac{\partial \phi^x(y)}{\partial \nu} - \phi^x(y)\frac{\partial u}{\partial \nu}(y))dS(y)$$

Using $\triangle_y \phi^x(y) = 0$ and $\phi^x(y) = \Phi(y-x) text\ when\ y \in \partial\Omega$, we have

$$-\int_\Omega \phi^x(y)\triangle u(y)dy$$

$$= \int_{\partial\Omega} (u(y)\frac{\partial \phi^x(y)}{\partial \nu} - \Phi(y-x)\frac{\partial u}{\partial \nu}(y))dS(y)$$

Combining $u(x) = \int_{\partial\Omega}(\Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi(y-x)}{\partial \nu})dS(y) - \int_\Omega \Phi(y-x)\Delta u(y)dy$
and

$$0 = \int_{\partial\Omega}(u(y)\frac{\partial \phi^x(y)}{\partial \nu} - \Phi(y-x)\frac{\partial u}{\partial \nu}(y))dS(y) + \int_\Omega \phi^x(y)\triangle u(y)dy,$$

we have $u(x) = \int_{\partial\Omega}(u(y)\frac{\partial(\phi^x(y)-\Phi(y-x))}{\partial \nu})dS(y) + \int_\Omega(\phi^x(y) - \Phi(y-x))\triangle u(y)dy$

Definition: Green's function for the region Ω is

$$G(x,y) = \Phi(y-x) - \phi^x(y), \quad (x, y \in \Omega, x \neq y)$$

where $\phi^x = \phi^x(y)$ solves $\begin{cases} \triangle_y \phi^x(y) = 0 & \text{in } \Omega \\ \phi^x(y) = \Phi(y-x) & \text{when } y \in \partial\Omega \end{cases}$

Note that $\triangle_y G(x,y) = \triangle(\Phi(y-z) - \phi^x(y)) = 0$ if $y \neq x$ and $G(x,y) = 0$ if $y \in \partial\Omega$.

Thus we have proved the following theorem.
Theorem (Representation formula using Green's function$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y))dS(y) - \int_{\Omega} G(x,y)\Delta u(y)dy$

Remark:
1. If $-\Delta u = f$ and $u|_{\partial\Omega} = g$ then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y))dS(y) + \int_{\Omega} G(x,y)f(y)dy.$$

2. If $\Delta u = 0$ and $u|_{\partial\Omega} = g$ then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y))dS(y).$$

Now we want to look at an important property of Green's function.

Theorem (Symmetry of Green's function For all $x, y \in \Omega$, $x \neq y$, we have

$$G(x, y) = G(y, x)$$

Proof: Fix $x, y \in \Omega$, $x \neq y$.
Write $v(z) = G(x, z)$ and $w(z) = G(y, z)$. Then v and w are smooth and harmonic in $\Omega \setminus \{x, y\}$.
Note that $\triangle_z v = \triangle_z w = 0$ and $v = w = 0$ on $\partial\Omega$.
Choose $\epsilon > 0$ small enough so that $B(x, \epsilon) \cap B(y, \epsilon) = \emptyset$, $B(x, \epsilon) \subset \Omega$ and $B(y, \epsilon) \subset \Omega$.
Now use Green's second identity in $\Omega \setminus (B(x, \epsilon) \cup B(y, \epsilon))$. We have $0 = \int_{\Omega \setminus (B(x,\epsilon) \cup B(y,\epsilon))} (v(z) \underbrace{\triangle_z w}_{=0} - w(z) \underbrace{\triangle_z v}_{=0}) dz$

$= \int_{\partial(\Omega \setminus (B(x,\epsilon) \cup B(y,\epsilon)))} (v(z) \frac{\partial w}{\partial \nu}(z) - w(z) \frac{\partial v}{\partial \nu}(z)) dS(z)$
$= \int_{\partial\Omega} \underbrace{v(z)}_{=0} \frac{\partial w}{\partial \nu}(z) - \underbrace{w(z)}_{=0} \frac{\partial v}{\partial \nu}(z)) dS(z)$
$+ \int_{\partial(B(x,\epsilon)} (v(z) \frac{\partial w}{\partial \nu}(z) - w(z) \frac{\partial v}{\partial \nu}(z)) dS(z)$
$+ \int_{\partial(B(y,\epsilon)} (v(z) \frac{\partial w}{\partial \nu}(z) - w(z) \frac{\partial v}{\partial \nu}(z)) dS(z)$

Thus we have $\int_{\partial B(x,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z)$
$+\int_{\partial B(y,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z) = 0.$

This implies that $\int_{\partial B(x,\epsilon)}(w(z)\frac{\partial v}{\partial \nu}(z) - v(z)\frac{\partial w}{\partial \nu}(z))dS(z)$
$= \int_{\partial B(y,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z)$. Recall that v is smooth at $\Omega \setminus \{x\}$ and w is smooth at $\Omega \setminus \{y\}$.

$$\int_{\partial B(x,\epsilon)}(w(z)\frac{\partial v}{\partial \nu}(z) - v(z)\frac{\partial w}{\partial \nu}(z))dS(z)$$
$$= \int_{\partial B(x,\epsilon)}(G(y,z)\frac{\partial G(x,z)}{\partial \nu} - \underbrace{G(x,z)}\frac{\partial G(y,z)}{\partial \nu})dS(z)$$
$$= \int_{\partial B(x,\epsilon)} \underbrace{G(y,z)}_{smooth\ \ near\ \ x} \frac{\partial}{\partial \nu}(\underbrace{\Phi(z-x)}_{\text{singular when z= x}} - \underbrace{\phi^x(z)}_{smooth\ \ near\ \ x})dS(z)$$
$$- \int_{\partial B(x,\epsilon)}(\underbrace{\Phi(z-x)}_{=C\epsilon^{2-n}} - \underbrace{\phi^y(x)}_{smooth\ \ function})\frac{\partial}{\partial \nu}(\underbrace{\Phi(z-y) - \phi^y(z)}_{smooth\ \ near\ \ x}))dS(z).$$

Note that $Area(\partial B(x,\epsilon)) = C\epsilon^{n-1}$. So we have
$lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)}(w(z)\frac{\partial v}{\partial \nu}(z) - v(z)\frac{\partial w}{\partial \nu}(z))dS(z)$
$= lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} \underbrace{G(y,z)}_{smooth \quad near \quad x} \frac{\partial}{\partial \nu}\Phi(z-x)dS(z)$

Recall that
$\int_{\partial B(x,\epsilon)} u(z)\frac{\partial}{\partial \nu}\Phi(z-x)dS(z) = \frac{1}{n\alpha_n\epsilon^{n-1}}\int_{\partial B(x,\epsilon)} u(z)dS(z)$ where u is smooth in $B(x,\epsilon)$.
(This is proved in the proof of Green's representation formula.)

Hence we have
$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} \underbrace{G(y,z)}_{smooth \quad near \quad x} \frac{\partial}{\partial \nu}\Phi(z-x)dS(z) = G(y,x)$. So
$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)}(w(z)\frac{\partial v}{\partial \nu}(z) - v(z)\frac{\partial w}{\partial \nu}(z))dS(z) = G(y,x)$.
Similarly,
$\lim_{\epsilon\to 0}\int_{\partial B(y,\epsilon)}(v(z)\frac{\partial w}{\partial \nu}(z) - w(z)\frac{\partial v}{\partial \nu}(z))dS(z) = G(x,y)$.
Thus we have $G(x,y) = G(y,x)$.