We'll prove that

$(W^{k,p}(\Omega), \| \ \|_{W^{k,p}(\Omega)})$ or

$((W^{k,p}(\Omega), \| \ \|'_{W^{k,p}(\Omega)}))$

is a Banach space

Recall that A Banach Space is a complete, normed linear space.

Th: The Sobolev space $(W^{k,p}(\Omega), \| \ \|_{W^{k,p}(\Omega)})$ or $(W^{k,p}(\Omega), \| \ \|'_{W^{k,p}(\Omega)})$ is a Banach Space where k is a positive integer and $1 \le p \le \infty$

pf:

1° From Th I (ii) on p247, We know $W^{k,p}(\Omega)$ is a linear space.

2° Now, we check that $\| \ \|_{W^{k,p}}$ or $\| \ \|'_{W^{k,p}}$ is a norm.

Recall that X is a Banach space if

(1) X is a linear space. (vector)

(2) X admits a norm, i.e. $\| \ \| : X \longrightarrow [0, \infty)$ s.t

(a) $\|u\| = 0 \Leftrightarrow u = 0$

(b) $\|\lambda u\| = |\lambda| \, \|u\|$

(c) $\|u+v\| \le \|u\| + \|v\|$

(3) Suppose $\{u_m\}_{m=1}^{\infty}$ is a Cauchy seq. Then we can find $u \in X$ s.t

$\lim_{m \to \infty} \|u_m - u\| = 0$

3° Recall $u \in W^{k,p}(\Omega)$

$$\|u\|_{W^{k,p}} = \Big(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}$$

$$\|u\|'_{W^{k,p}} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}$$

If $\|u\|_{W^{k,p}} = 0 \Rightarrow \sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)} = 0$

Note that $\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p}$

$$= \|u\|^p_{L^p(\Omega)} + \sum_{1 \le |\alpha| \le k} \|D^\alpha u\|^p_{L^p}$$

So $\|u\|_{W^{k,p}} = 0 \Rightarrow \|u\|_{L^p(\Omega)} = 0$

$\Rightarrow u = 0$ in $L_p(\Omega)$

Also $u = 0$ is in $W^{k,p}(\Omega)$

$$\|u\|'_{W^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)}$$

$$= \|u\|_{L^p(\Omega)} + \sum_{1\le|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)}$$

$$\|u\|'_{W^{k,p}} = 0 \Rightarrow \|u\|_{L^p(\Omega)} = 0$$

$$\Rightarrow u = 0 \text{ in } W^{k,p}(\Omega)$$

4° Now want to show that

$$\|\lambda u\|_{W^{k,p}} = |\lambda|\, \|u\|_{W^{k,p}}$$

and $\|\lambda u\|'_{W^{k,p}} = |\lambda|\, \|u\|'_{W^{k,p}}$

Since $\|\lambda u\|_{W^{k,p}} = \left(\sum_{|\alpha|\le k} \|D^\alpha(\lambda u)\|^p_{L^p}\right)^{\frac{1}{p}}$

$$= |\lambda|\, \|u\|_{W^{k,p}}$$

Similarly, $\|\lambda u\|'_{W^{k,p}} = \sum_{|\alpha|\le k} \overbrace{\|D^\alpha(\lambda u)\|_{L^p(\Omega)}}^{\|\lambda D^\alpha(u)\|_{L^p}}$

$$= |\lambda| \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)} = |\lambda|\, \|u\|'_{W^{k,p}}$$

5° Now want to prove that

$\|u+v\|_{W^{k,p}} \le \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}$ and

$$\|u+v\|'_{W^{k,p}} \le \|u\|'_{W^{k,p}} + \|v\|'_{W^{k,p}}$$

Recall Minkowski's inequality

(i) $\|u+v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}$

(ii) $\left(\sum_i (a_i+b_i)^p\right)^{\frac{1}{p}} \le \left(\sum_i a_i^p\right)^{\frac{1}{p}} + \left(\sum_i b_i^p\right)^{\frac{1}{p}}$

$$\|u+v\|'_{W^{k,p}(\Omega)} = \sum_{|\alpha|\le k} \|D^\alpha(u+v)\|_{L^p(\Omega)}$$

$$= \sum_{|\alpha|\le k} \|\underbrace{D^\alpha u}_{L^p} + \underbrace{D^\alpha v}_{L^p}\|_{L^p(\Omega)}$$

$$\underset{\text{M ineq.}}{\le} \sum_{|\alpha|\le k} \left(\|D^\alpha u\|_{L^p(\Omega)} + \|D^\alpha u\|_{L^p(\Omega)}\right)$$

$$= \|u\|'_{W^{k,p}} + \|v\|'_{W^{k,p}}$$

$$\|u+v\|_{W^{k,p}} = \left(\sum_{|\alpha|\le k} \|D^\alpha(u+v)\|^p_{L^p}\right)^{\frac{1}{p}}$$

$$\le \left(\sum_{|\alpha|\le k} \left(\|D^\alpha(u)\|_{L^p} + \|D^\alpha v\|_{L^p}\right)^p\right)^{\frac{1}{p}}$$

$$\underset{\substack{\text{Discrete}\\ \text{M-ineq.}}}{\le} \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p\right)^{\frac{1}{p}} + \left(\sum_{|\alpha|\le k} \|D^\alpha v\|^p\right)^{\frac{1}{p}}$$

$$= \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}$$

6° It remains to show that $W^{k,p}$ is complete.

Since $\|\ \|'$ and $\|\ \|'$ are equivalent, it suffices to show that $(W^{k,p}, \|\ \|')$ is complete.

Assume $\{u_m\}_{m=1}^{\infty}$ is a cauchy seq in $W^{k,p}$ w.r.t $\| \ \|'_{W^{k,p}}$.

Since $\|u_n - u_m\|'_{W^{k,p}}$

$$= \|u_n - u_m\|_{L^p(\Omega)} + \sum_{1\le|\alpha|\le k} \|D^\alpha u_n - D^\alpha u_m\|_{L^p(\Omega)},$$

We know that $\{u_n\}_{n=1}^{\infty}$ is a cauchy seq in L_p and $\{D^\alpha u_n\}_{n=1}^{\infty}$ is also a cauchy seq in L_p.

b/c L_p is a Banach space.

$\Rightarrow$ We can find $u \in L^p$, $u^\alpha \in L^p$ for $|\alpha| \le k$

s.t $\lim_{n\to\infty} \|u_n - u\|_{L^p} = 0$ and

$\lim_{n\to\infty} \|D^\alpha u_n - u^\alpha\|_{L^p} = 0$

Now we want to show that $u \in W^{k,p}(\Omega)$ and $D^\alpha u = u^\alpha$

This implies that $\lim_{n\to\infty} \|u_n - u\|_{W^{k,p}} = 0$

$$\left(\lim_{n\to\infty} \|u_n - u\|_{L^p} + \sum_{|\alpha|\le k} \|D^\alpha u_n - \overset{\overset{u^\alpha}{\|}}{D^\alpha u}\|_{L_p} \right.$$

$\Rightarrow W^{k,p}$ is a Banach space

Given $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega u D^\alpha \phi \, dx = \lim_{m\to\infty} \int u_m D^\alpha \phi \quad \left(\begin{array}{l} b/c\ D^\alpha \phi \in L^q(\Omega) \\ \text{when } \frac{1}{p}+\frac{1}{q}=1 \end{array} \right)$$

$$\left(b/c \int |u D^\alpha\phi - u_m D^\alpha \phi| \right.$$

$$\le \left(\int \underset{L^p}{\underset{\uparrow}{|u-u_m|}} \, \underset{L^q}{\underset{\uparrow}{|D^\alpha\phi|}} \right) \underset{\text{Holder ineq}}{\underset{\uparrow}{\le}} \left(\int |u-u_m|^p \right)^{\frac{1}{p}} \left(\int (D^\alpha\phi)^q \right)^{\frac{1}{q}} \Bigg)$$

$$\left(\int |u-u_m|^p \right)^{\frac{1}{p}} \downarrow 0$$

$$= \lim_{m\to\infty} (-1)^{|\alpha|} \int_\Omega \underset{\underset{u^\alpha}{\downarrow \text{ in } L_p}}{D^\alpha u_m} \phi \quad \left(\begin{array}{l} b/c\ u_m \in W^{k,p} \\ \Rightarrow D^\alpha u_m \text{ exists} \end{array} \right)$$

$$= (-1)^{|\alpha|} \int_\Omega u^\alpha \phi$$

$$\Rightarrow \int_\Omega u D^\alpha \phi = (-1)^{|\alpha|} \int_\Omega \underline{\underline{u^\alpha}} \phi$$

$\Rightarrow D^\alpha u = u^\alpha$

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