PDE NOTES on Sep 29

(pf of Green's representation formula)

Given $x \in \Omega$,

Choose $\varepsilon > 0$ small enough so that

$$B(x,\varepsilon) \subset \Omega$$

Note that $\Phi(y-x)$ is not defined when $y = x$.

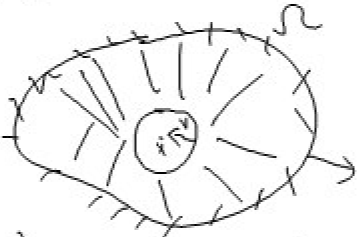
Apply Green's 2-nd Identity to $u(y)\,\Phi(y-x)$ (regard this as a fth of y) in $\Omega \setminus B(x,\varepsilon)$

$$\int_{\Omega \setminus B(x,\varepsilon)} \left(u(y)\,\Delta_y \Phi(y-x) - \Phi(y-x)\,\Delta_y u(y) \right) dy$$

$$= \int_{\partial(\Omega \setminus B(x,\varepsilon))} \left[u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \right] dS(y)$$

Recall that Φ is a fundamental sol

$$\Rightarrow \quad \Delta_y \Phi(y-x) = 0$$



$$\partial(\Omega \setminus B(x,\varepsilon)) = \partial\Omega \cup \partial(B(x,\varepsilon))$$

$$\Rightarrow \int_{\partial(\Omega \setminus B(x,\varepsilon))} \left[u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \right] dS(y)$$

$$= \int_{\partial\Omega} \left[u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \right] dS(y)$$

$$+ \int_{\partial(B(x,\varepsilon))} \left[u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u(y)}{\partial \nu} \right] dS(y)$$

Claim $\displaystyle\int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi(y-x)}{\partial \nu}\,dS(y) = \frac{1}{n\alpha_n \varepsilon^{n-1}}\int_{\partial B(x,\varepsilon)} u(y)\,dS(y)$

Recall that $\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| \\ \frac{1}{n(n-2)\alpha_n}|x|^{2-n} \end{cases}$

$$\Rightarrow \nabla\Phi(x) = \begin{cases} -\frac{1}{2\pi|x|}\frac{x}{|x|} \\ -\frac{1}{n\alpha_n}|x|^{1-n}\frac{x}{|x|} \end{cases}$$

$$\Rightarrow \nabla_y\Phi(y-x) = \begin{cases} -\frac{1}{2\pi|y-x|}\frac{y-x}{|y-x|} \\ -\frac{1}{n\alpha_n}|y-x|^{1-n}\frac{y-x}{|y-x|} \end{cases}$$

On $\partial B(x,\varepsilon)$

$\nu = -\frac{(y-x)}{\|y-x\|}$

$$\Rightarrow \frac{\partial\Phi(y-x)}{\partial\nu} = \nabla_y\Phi(y-x)\cdot\nu$$

$$= \begin{cases} -\frac{1}{2\pi|y-x|}\cdot\frac{y-x}{|y-x|}\cdot\left(-\frac{(y-x)}{|y-x|}\right) & n=2 \\ -\frac{1}{n\alpha_n}|y-x|^{1-n}\frac{y-x}{|y-x|}\cdot\left(-\frac{(y-x)}{|y-x|}\right) & n\geq 3 \end{cases}$$

$$= \begin{cases} \frac{1}{2\pi|y-x|} & n=2 \\ \frac{1}{n\alpha_n|y-x|^{n-1}} & n\geq 3 \end{cases}$$

$y\in\partial B(x,\varepsilon) \Rightarrow \|y-x\| = \varepsilon$

$$\Rightarrow \frac{\partial\Phi(y-x)}{\partial\nu} = \begin{cases} \frac{1}{2\pi\varepsilon} & n=2 \\ \frac{1}{n\alpha_n\varepsilon^{n-1}} & n\geq 3 \end{cases}$$

$$\Rightarrow \int_{\partial B(x,\varepsilon)} u(y)\frac{\partial\Phi(y-x)}{\partial\nu}\,dS(y) = \frac{1}{n\alpha_n\varepsilon^{n-1}}\int_{\partial B(x,\varepsilon)} u(y)\,dS(y)$$

Claim: $\displaystyle\left|\int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial u}{\partial\nu}\,dS(y)\right| \leq \begin{cases} C\varepsilon|\ln\varepsilon| & n=2 \\ C\varepsilon & n\geq 3 \end{cases}$

Note that $|\Phi(y-x)| = \begin{cases} C|\ln\varepsilon| \\ \frac{C}{\varepsilon^{n-2}} \end{cases}$ where $y\in\partial B(x,\varepsilon)$ $(|y-x|=\varepsilon)$

b/c $u\in C^2(\bar\Omega) \Rightarrow |Du|\leq C$ in $\bar\Omega$

$$\left|\frac{\partial u}{\partial \nu}\right| = |\nabla_y u \cdot \nu| \le |\nabla_y u| \cdot |\nu|$$
$$\le C \quad (b/c\ |\nu| = 1)$$

$$\Rightarrow \left| \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu} \, dS(y) \right|$$

$$\le \int_{\partial B(x,\varepsilon)} |\Phi(y-x)| \left|\frac{\partial u}{\partial \nu}\right| dS(y)$$

$$\le C \int_{\partial B(x,\varepsilon)} |\Phi(y-x)| \, dS(y)$$

$$\le C \begin{cases} \varepsilon |\ln \varepsilon| & n=2 \\ \varepsilon^{n-1} \cdot \frac{1}{\varepsilon^{n-2}} & n \ge 3 \end{cases}$$

$$= \begin{cases} C \varepsilon |\ln \varepsilon| & n=2 \\ C \varepsilon & n \ge 3 \end{cases}$$

Combining previous two claims:

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} u(y) \frac{\partial \Phi(y-x)}{\partial \nu} dS(y)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{n \alpha_n \varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} u(y) \, dS(y)$$

$$= u(x) \quad (b/c\ u \text{ is cts})$$

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u(y)}{\partial \nu} dS(y)$$

$$= 0$$

Let $\varepsilon \longrightarrow 0$

$$\int_\Omega -\Phi(y-x) \Delta_y u(y) \, dy$$

$$= u(x) + \int_{\partial\Omega} \left[u(y) \frac{\partial \Phi(y-x)}{\partial \nu} - \Phi(y-x) \frac{\partial u}{\partial \nu} \right] dS(y)$$

$$u(x) = \int \left[\Phi(y-x) \frac{\partial u}{\partial \nu} - u(y) \frac{\partial \Phi(y-x)}{\partial \nu} \right] dS(y) - \int_\Omega \Phi(y-x) \Delta u(y) \, dy$$

↑ Green's representation formula.