PDE Notes on Sep 26, 2008

Recall
$$\begin{cases} v = \ln u \\ w = |\nabla v|^2 = \dfrac{|\nabla u|^2}{u^2} \geq 0 \\ \phi : B_2 \longrightarrow \mathbb{R} \\ \phi > 0 \text{ in } B_2 \text{ and } \phi\big|_{\partial B_2} = 0 \end{cases}$$

From last time, we derived

$$\Delta(w\phi^4) + 2\nabla v \cdot \nabla(w\phi^4)$$
$$\geq \frac{\phi^4 w^2}{n} - c_1\phi^2 w - c_2\phi^2 w - c_3\phi^3 w^{\frac{3}{2}}$$
$$\left[\text{where } c_1, c_2, c_3 \text{ depends on } \phi \text{ (not on } w).\right]$$
$$\phi^2 w\left[\frac{\phi^2 w}{n} - c_3\phi w^{\frac{1}{2}} - (c_1+c_2)\right]$$

Since
$$\begin{cases} w\phi^4 \geq 0 \text{ in } B_2 \\ w\phi^4\big|_{\partial B_2} = 0 \quad (b/c\ \phi\big|_{\partial B_2} = 0), \end{cases}$$

We can find $x_0 \in B_2$ s.t.
$$(w\phi^4)(x_0) = \max_{B_2} w\phi^4$$

$$\Rightarrow \begin{cases} \nabla(w\phi^4)(x_0) = 0 \\ \Delta(w\phi^4)(x_0) \leq 0 \end{cases}$$

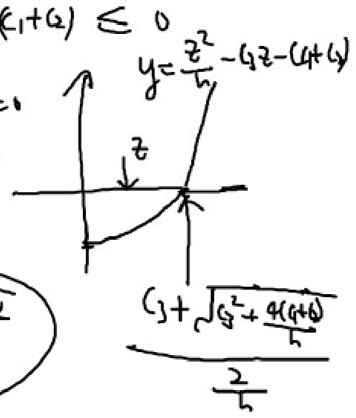
$$\Rightarrow \quad \phi^2 w(x_0)\left[\frac{\phi^2 w(x_0)}{n} - c_3\phi w^{\frac{1}{2}}(x_0) - (c_1+c_2)\right] \leq 0$$

$$\Rightarrow \quad \text{Let } z = (\phi w^{\frac{1}{2}})(x_0) \geq 0 \quad b/c\ \phi^2 w(x_0) \geq 0$$

$$\Rightarrow \quad \frac{z^2}{n} - c_3 z - (c_1+c_2) \leq 0$$

The roots of $\frac{x^2}{n} - c_3 x - (c_1+c_2) = 0$

are $x = \dfrac{c_3 \pm \sqrt{c_3^2 + \frac{4(c_1+c_2)}{n}}}{\frac{2}{n}}$

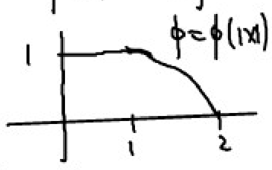


$$z \leq \underbrace{\frac{c_3 + \sqrt{c_3^2 + \frac{4(c_1+c_2)}{n}}}{\frac{2}{n}}}_{= C(n)}$$

$$\Rightarrow \quad (\phi^2 w)(x_0) \leq C(n)$$

where $\phi^4 w(x_0) = \sup\limits_{B_2} \phi^4 w$

Require $0 \leq \phi \leq 1$ and $\phi \equiv 1$ in B_1



$$\sup_{B_2} \phi^4 w = \phi^4 w(x_0)$$
$$\leq \phi^2 w(x_0) \quad (b/c\ 0 \leq \phi \leq 1)$$
$$\leq C(n)$$

Moreover,
$$\underbrace{\sup_{B_1} \phi^4 w}_{= \sup_{B_1} w} \leq \sup_{B_2} \phi^4 w \leq C(n)$$
$$(b/c\ \phi \equiv 1 \text{ in } B_1)$$

Recall that $w = \dfrac{|\nabla u|^2}{u^2}$

$$\Rightarrow \quad \sup_{B_1} \frac{|\nabla u|^2}{u^2} \leq C(n)$$

$$\Rightarrow \quad \sup_{B_1} \frac{|\nabla u|}{u} \leq \sqrt{C(n)}$$

So we have proved that

$$\begin{cases} u: B_2 \to \mathbb{R} \text{ and harmonic} \\ u > 0 \end{cases}$$

$$\Rightarrow \quad \frac{|\nabla u|}{u} \le C(n) \text{ in } B_1$$

Now we consider the general case.

$u: B_{2r} \to \mathbb{R}$, $u > 0$, harmonic.

Let $v(x) = u(rx)$.

Then $v: B_2 \to \mathbb{R}$, $v > 0$, harmonic

$$\Rightarrow \quad \frac{|\nabla v(x)|}{v(x)} \le C(n), \ x \in B_1$$

Compute $\nabla v(x) = r \nabla u(rx)$

$$\Rightarrow \quad \frac{|\nabla v(x)|}{v(x)} = \frac{r|\nabla u(rx)|}{u(rx)} \le C(n)$$

$$\Rightarrow \quad \frac{|\nabla u(rx)|}{u(rx)} \le \frac{C(n)}{r}$$

$$\Rightarrow \quad \frac{|\nabla u(y)|}{u(y)} \le \frac{C(n)}{r}, \ y \in B_r$$

The techniques used in this proof can be used to study other PDE.

For example: $\Delta u = 0$ on a Riemann mfld with $Rc \ge 0$

Minimal surface eqs.

Gradient estimate of this type $\left(\frac{|\nabla u|}{u} \le \frac{C}{r} \right)$ holds for other eqs.

$\Downarrow$ This implies

Harnack inequality.

§2.2.4 Green's fcn

Assume $\Omega \subset \mathbb{R}^n$ is open, bounded and $\partial\Omega$ is C^1 (so the normal is well-defined)

In the following, we'll derive a representation formula for the sol of $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

Th: (Green's representation formula)

$u \in C^2(\bar{\Omega})$

$$\Rightarrow u(x) = \int_{\partial\Omega} \left(\Phi(y-x) \frac{\partial u(y)}{\partial \nu} - u(y) \frac{\partial \Phi(y-x)}{\partial \nu} \right) dS(y) - \int_{\Omega} \Phi(y-x) \Delta u(y) dy$$

for any $x \in \Omega$

where $\underset{\substack{\uparrow \\ \text{fundamental sol}}}{\Phi(x) =} \begin{cases} -\frac{1}{2\pi} \ln|x| \\ \frac{1}{n(n-2)\alpha_n} |x|^{2-n} \end{cases}$

Remark ① If $\Delta u = 0$ in Ω

$$\Rightarrow u(x) = \int_{\partial\Omega} \left(\Phi(y-x) \frac{\partial u(y)}{\partial \nu} - \underline{u(y)} \frac{\partial \Phi(y-x)}{\partial \nu} \right) dS(y)$$

Disadvantage: This formula depends on $\frac{\partial u}{\partial \nu}$

This is unknown.

Boundary value (known)