Solutions to HW 2

(1) (Problem 2.1 from Lec2) (15 pts)

If A is triangular, then A is either upper-triangular or lower triangular. Let us assume A is upper-triangular first. A is unitary then $A^{-1} = A^*$. Since A is upper-triangular, we know that A^{-1} is upper-triangular by exercise 1.3 and A^* is lower-triangular. In particular, A^* is diagonal and A is diagonal.

The case that A is lower-triangular can be proved in a similar way.

(2) (Problem 2.3 from Lec2) In the following, we use the notation $x^*y = \langle x, y\rangle$.

(a) (10 pts) Recall that $\langle x, Ax\rangle = \langle A^*x, x\rangle$. Using $Ax = \lambda x$ and $A = A^*$, we get $\lambda|x|^2 = \overline{\lambda}|x|^2$. Thus we have $\lambda = \overline{\lambda}$, that is, λ is real.

(b) (10 pts) Suppose $Ax = \lambda x$ and $Ay = \mu y$ where $\lambda \in R$, $\mu \in R$ and $\lambda \neq \mu$. Then $\lambda\langle x, y\rangle = \langle Ax, y\rangle = \langle x, A^*y\rangle = \langle x, Ay\rangle = \mu\langle x, y\rangle$. Thus $\lambda\langle x, y\rangle = \mu\langle x, y\rangle$. Since $\lambda \neq \mu$, we must have $\langle x, y\rangle = 0$. Thus x and y are orthogonal.

(3) (Problem 2.4 from Lec2) (10 pts) Suppose λ is an eigenvalues of A with eigenvector nonzero eigenvector x. Then $Ax = \lambda x$. Since A is unitary, we have $\langle x, x\rangle = \langle Ax, Ax\rangle = \langle \lambda x, \lambda x\rangle = |\lambda|^2\langle x, x\rangle$. Therefore $|\lambda|^2 = 1$. This means that the absolute value of any eigenvalue of a unitary matrix is one.

(4) (Problem 2.5 from Lec2)

(a)(10 pts) Since $(iS)^* = -iS^* = -i(-S) = iS$, we have iS is hermitian.

So the eigenvalues of iS are real and the eigenvalues of S are pure imaginary. We have used that fact that if λ is an eigenvalue of S then $i\lambda$ is an eigenvalue of iS.

(b) (10 pts) We prove it by contradiction. Suppose $I - S$ is singular. This implies that 1 is an eigenvalue of S which contradicts to the fact that the eigenvalue of S is pure imaginary.

(c) (15 pts) Let $Q = (I - S)^{-1}(I + S)$. Then $Q^* = \left((I - S)^{-1}(I + S)\right)^* = (I + S)^*(I - S)^{-1*} = (I + S)^*(I - S)^{*-1} = (I - S)(I + S)^{-1}$. Then $QQ^* = (I - S)^{-1}(I + S)(I - S)(I + S)^{-1} = (I - S)^{-1}(I - S)(I + S)(I + S)^{-1} = I$. We have used the fact that $(I + S)(I - S) = (I - S)(I + S) = I - S^2$.

(5) (Problem 2.6 from Lec2) (20 pts) We compute

$$
\begin{aligned}
(I + \alpha uv^*)(I + uv^*) &= (I + uv^*) + \alpha uv^* + \alpha uv^*uv^* \\
&= I + (1 + \alpha + \alpha v^*u)uv^*
\end{aligned}
$$

Note that v^*u is a scalar(number). If $1 + v^*u \neq 0$ then we can choose α such that $1 + \alpha + \alpha v^*u = 0$, i.e. $\alpha = -\frac{1}{1+v^*u}$. This implies that A is nonsingular if $1 + v^*u \neq 0$ with $A^{-1} = I - \frac{1}{1+v^*u}uv^*$

If $1 + v^*u = 0$ then A is singular. Suppose $w \in Null(A)$. Then $(I + uv^*)w = 0$. This implies that $w + (v^*w)u = 0$ and $w = -(v^*w)u$. Thus we know that $Null(A) = span(u)$.