Linear Algebra (Math 2890) Review Problems for Final Exam

Final exam on May 5 (Wednesday) 5 pm-7 pm.
Regular office hours:
UH2080B M 12-2 pm, W 1-2, 4-5pm, F 1-2 pm or make appointment
Office hour before the final exam:
Monday (May 3) 12-2 pm, Tuesday (May 4) 12-2pm
Wednesday (May 5) 12-2 pm

Topics in the final exam. The final exam is compressive. It covers 1.1-1.5,
1.7, 1.8, 2.1-2.3, 2.8, 2.9, 3.1, 3.2, 5.1-5.3, 6.1-6.6, 7.1, 7.2.

1. Let $A$ be the matrix

$$
\begin{bmatrix}
  2 & 1 & 1 \\
  1 & 2 & 1 \\
  1 & 1 & 2 \\
\end{bmatrix}
$$

(a) Prove that $\det(A - \lambda I) = - (\lambda - 1)^2(\lambda - 4)$.  
(b) Find the eigenvalues and a basis of eigenvectors for $A$.  
(c) Diagonalize the matrix $A$ if possible.  
(d) Find an expression for $A^k$. (e) Find an expression for the matrix exponential $e^A$.

2. Let $B$ be the matrix

$$
\begin{bmatrix}
  2 & 1 & 1 \\
  0 & 2 & 1 \\
  0 & 0 & 1 \\
\end{bmatrix}
$$

(a) Find the characteristic equation of $B$.  
(b) Find the eigenvalues and a basis of eigenvectors for $B$.  
(c) Diagonalize the matrix $B$ if possible.

3. Let $A$ be the matrix

$$
A = \begin{bmatrix}
-4 & -5 & 5 \\
-5 & -4 & -5 \\
5 & -5 & -4 \\
\end{bmatrix}
$$

(a) Prove that $\det(A - \lambda I) = (9 + \lambda)^2(6 - \lambda)$. You may use the fact that $(9 + \lambda)^2(6 - \lambda) = 486 + 27\lambda - 12\lambda^2 - \lambda^3$.  
(b) Orthogonally diagonalizes the matrix $A$, giving an orthogonal matrix $P$ and a diagonal matrix $D$ such that $A = PDP^t$.  

1
(c) Write the quadratic form associated with $A$ using variables $x_1$, $x_2$, and $x_3$?

(d) Find an expression for $A^k$ and $e^A$.

(e) What’s $A^3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$?

(f) What is $\lim_{n \to \infty} A^{-n} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$?

4. Classify the quadratic forms for the following quadratic forms. Make a change of variable $x = Py$, that transforms the quadratic form into one with no cross term. Also write the new quadratic form in new variables $y_1, y_2$.

(a) $9x_1^2 - 8x_1x_2 + 3x_2^2$.
(b) $-5x_1^2 + 4x_1x_2 - 2x_2^2$.
(c) $8x_1^2 + 6x_1x_2$.

5. (a) Find a $3 \times 3$ matrix $A$ which is not diagonalizable?
(b) Give an example of a $2 \times 2$ matrix which is diagonalizable but not orthogonally diagonalizable?

6. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$.

(a) Find the condition on $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ such that $Ax = b$ is consistent.

(b) What is the column space of $A$?
(c) Describe the subspace $\text{col}(A)^\perp$ and find an basis for $\text{col}(A)^\perp$. What’s the dimension of $\text{col}(A)^\perp$?

(d) Use Gram-Schmidt process to find an orthogonal basis for the column space of $A$.

(e) Find an orthonormal basis for the column of the matrix $A$.

(f) Find the orthogonal projection of $y = \begin{bmatrix} 7 \\ 3 \\ 10 \\ -2 \end{bmatrix}$ onto the column space of $A$ and write $y = \hat{y} + z$ where $\hat{y} \in \text{col}(A)$ and $z \in \text{col}(A)^\perp$. Also find the shortest distance from $y$ to $\text{Col}(A)$.

(g) Using previous result to explain why $Ax = y$ has no solution.

(h) Use orthogonal projection to find the least square solution of $Ax = y$.

(i) Use normal equation to find the least square solution of $Ax = y$.

7. Find the equation $y = a + mx$ of the least square line that best fits the given data points. $(0,1), (1,1), (3,2)$.

8. (a) Let $A = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}$. Find the inverse matrix of $A$ if possible.

(b) Find the coordinates of the vector $(1, -1, 2)$ with respect to the basis $B$ obtained from the column vectors of $A$.

9. Let $H = \left\{ \begin{bmatrix} a + 2b - c \\ a - b - 4c \\ a + b - 2c \end{bmatrix} : a, b, c \text{ any real numbers} \right\}$.

a. Explain why $H$ is a subspace of $\mathbb{R}^3$.

b. Find a set of vectors that spans $H$.

c. Find a basis for $H$.

d. What is the dimension of the subspace?

e. Find an orthogonal basis for $H$. 
10. Determine if the following systems are consistent and if so give all solutions in parametric vector form.

(a) 
\[
\begin{align*}
-2x_2 &= 3 \\
2x_1 - 7x_2 &= 0 \\
-5x_1 + 8x_2 &= 5
\end{align*}
\]

(b) 
\[
\begin{align*}
x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\
x_1 - 2x_2 + 4x_3 - x_4 &= 6 \\
-2x_1 - 4x_2 + 7x_3 - x_4 &= 1
\end{align*}
\]

11. Let 
\[
A = \begin{bmatrix}
1 & -3 & 4 & -2 & 5 \\
2 & -6 & 9 & -1 & 8 \\
2 & -6 & 9 & -1 & 9 \\
-1 & 3 & -4 & 2 & -5
\end{bmatrix}
\]
which is row reduced to 
\[
\begin{bmatrix}
1 & -3 & -20 & -3 \\
0 & 0 & 1 & 3 & 3 \\
0 & 0 & 1 & 3 & 4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) Find a basis for the column space of \( A \)
(b) Find a basis for the nullspace of \( A \)
(c) Find the rank of the matrix \( A \)
(d) Find the dimension of the nullspace of \( A \).

(e) Is 
\[
\begin{bmatrix}
1 \\
4 \\
3 \\
1
\end{bmatrix}
\]
in the range of \( A \)?

(e) Does \( Ax = 
\begin{bmatrix}
0 \\
3 \\
2 \\
0
\end{bmatrix}
\]
have any solution? Find a solution if it’s solvable.

12. Determine if the columns of the matrix form a linearly independent set. Justify your answer.
\[
\begin{bmatrix}
1 & 1 \\
1 & 2
\end{bmatrix}, \quad \begin{bmatrix}
1 & -2 \\
-2 & 4 \\
3 & 6
\end{bmatrix}, \quad \begin{bmatrix}
-4 & -3 & 0 \\
0 & -1 & 4 \\
1 & 0 & 3 \\
5 & 4 & 6
\end{bmatrix}, \quad \begin{bmatrix}
-4 & -3 & 1 & 5 & 1 \\
2 & -1 & 4 & -1 & 2 \\
1 & 2 & 3 & 6 & -3 \\
5 & 4 & 6 & -3 & 2
\end{bmatrix}
\]