6.4 Gram-Schmidt Process

In this section, we will learn a process for constructing an orthogonal (orthonormal) basis for subspace W of R^n. Recall that given am orthogonal set $\{u_1, u_2, \cdots, u_p\}$. Then we can normalize it to get an orthonormal set $\{\frac{u_1}{||u_1||}, \frac{u_2}{||u_2||}, \cdots, \frac{u_p}{||u_p||}\}$.

First, let us recall that if $\{u_1, u_2, \cdots, u_p\}$ is an orthogonal set. Then $y - \frac{y\cdot u_1}{u_1\cdot u_1}u_1 - \frac{y\cdot u_2}{u_2\cdot u_2}u_2 - \cdots - \frac{y\cdot u_p}{u_p\cdot u_p}u_p$ is orthogonal to $Span\{u_1, u_2, \cdots, u_p\}$.

Let start with two vectors $W = Span\{u_1, u_2\}$ and $\{u_1, u_2\}$ is a basis for W. How can we construct an orthogonal basis out of $\{u_1, u_2\}$?

First, let $v_1 = u_1$. Let $v_2 = u_2 - \frac{u_2\cdot v_1}{v_1\cdot v_1}v_1$. By orthogonal projection theorem, we know that v_2 is orthogonal to v_1. Then $\{v_1, v_2\}$ is an orthogonal basis for W and and $\{\frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}\}$ is an orthonormal basis for subspace W.

Now let us look at the case of three vectors $W = Span\{u_1, u_2, v_3\}$ and $\{u_1, u_2, u_3\}$ is a basis for W.

Step1. Let $v_1 = u_1$.

Step 2. Let $v_2 = u_2 - \frac{u_2\cdot v_1}{v_1\cdot v_1}v_1$.

Now $\{v_1, v_2\}$ is is an orthogonal set.

Step 3. Let $v_3 = u_3 - Proj_{Span\{v_1,v_2\}}(u_3) = u_3 - \frac{u_3\cdot v_1}{v_1\cdot v_1}v_1 - \frac{u_3\cdot v_2}{v_2\cdot v_2}v_2$.

Then $\{v_1, v_2, v_3\}$ is an orthogonal basis for W and $\{\frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \frac{v_3}{||v_3||}\}$ is an orthonormal basis for subspace W.

In general, given a subspace $W = Span\{u_1, u_2, \cdots, u_p\}$ where $\{u_1, u_2, \cdots, u_p\}$ is a basis for W. Then $v_1 = u_1$, $u_2 - \frac{u_2\cdot v_1}{v_1\cdot v_1}v_1$, $v_3 = u_3 - \frac{u_3\cdot v_1}{v_1\cdot v_1}v_1 - \frac{u_3\cdot v_2}{v_2\cdot v_2}v_2$, $\cdots$, $v_p = u_p\frac{u_p\cdot v_1}{v_1\cdot v_1}v_1 - \frac{u_p\cdot v_2}{v_2\cdot v_2}v_2 - \cdots - \frac{u_p\cdot v_{p-1}}{v_{p-1}\cdot v_{p-1}}v_{p-1}$ gives us a orthogonal basis $\{v_1, v_2, \cdots, v_p\}$. The is called the Gram-Schmidt process.

Example 1 *(a) Find an orthogonal basis and an orthonormal basis for the column space of the following matrix.*

$$A = \begin{bmatrix} -1 & 6 & 5 & 6 \\ 3 & -8 & -5 & 3 \\ 1 & -2 & -1 & 6 \\ 1 & -4 & -3 & -3 \end{bmatrix}.$$

(b) Find the distance between the point $y = \begin{bmatrix} -10 \\ 7 \\ -4 \\ 3 \end{bmatrix}$ *and* $Col(A)$

Solution: First, we find a basis for $Col(A)$ by row reduction.

$$A = \begin{bmatrix} -1 & 6 & 5 & 6 \\ 3 & -8 & -5 & 3 \\ 1 & -2 & -1 & 6 \\ 1 & -4 & -3 & -3 \end{bmatrix} \sim \quad r_1 \longleftrightarrow r_3 \begin{bmatrix} 1 & -2 & -1 & 6 \\ 3 & -8 & -5 & 3 \\ -1 & 6 & 5 & 6 \\ 1 & -4 & -3 & -3 \end{bmatrix}$$

$$\sim r_2 := r_2 - 3r_1, r_3 := r_3 + r_1, r_4 := r_4 - r_1 \begin{bmatrix} 1 & -2 & -1 & 6 \\ 0 & -2 & -2 & -15 \\ 0 & 4 & 4 & 12 \\ 0 & -2 & -2 & -9 \end{bmatrix}$$

$$\sim r_3 := r_3/4 \begin{bmatrix} 1 & -2 & -1 & 6 \\ 0 & -2 & -2 & -15 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -9 \end{bmatrix} \sim r_2 \longleftrightarrow r_3 \begin{bmatrix} 1 & -2 & -1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & -2 & -15 \\ 0 & -2 & -2 & -9 \end{bmatrix}$$

$$\sim r_3 := r_3 + 2r_2, r_4 := r_4 + 2r_2 \begin{bmatrix} 1 & -2 & -1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\sim r_3 := \frac{r_3}{-9}, r_4 := r_4 + 3r_3 \begin{bmatrix} 1 & -2 & -1 & -7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we know the first, second and the forth column of A is a basis for the column space of A. Let $u_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$.

Then $Col(A) = Span\{u_1, u_2, u_3\}$.
Now we can use the formula

$$v_1 = u_1,$$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

and

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

to get an orthogonal basis.

Let $v_1 = u_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$.

$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$. Compute $u_2 \cdot v_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = -6 - 24 - 2 - 4 = -36$

and

$v_1 \cdot v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 1 + 1 + 9 + 1 = 12.$

So $v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - (\frac{-36}{12}) \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} =$

$\begin{bmatrix} 6-3 \\ -8+9 \\ -2+3 \\ -4+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$.

Compute $u_3 \cdot v_1 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = -6 + 9 + 6 - 3 = 6,$

$v_1 \cdot v_1 = 12$ (We have already computed it earlier),

$u_3 \cdot v_2 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 18 + 3 + 6 + 3 = 30,$

$v_2 \cdot v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 9 + 1 + 1 + 1 = 12.$

So

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$
$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad (0.1)$$
$$= \begin{bmatrix} 6 + \frac{1}{2} - \frac{15}{2} \\ 3 - \frac{3}{2} - \frac{5}{2} \\ 6 - \frac{1}{2} - \frac{5}{2} \\ -3 - \frac{1}{2} + \frac{5}{2} \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Hence $\{v_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}\}$ is an orthogonal basis for $Col(A)$. We can find that $||v_1|| = \sqrt{v_1 \cdot v_1} = \sqrt{12} = 2\sqrt{3}$, $||v_2|| = \sqrt{v_2 \cdot v_2} = \sqrt{12} = 2\sqrt{3}$ and $||v_3|| = \sqrt{v_3 \cdot v_3} = \sqrt{12} = 2\sqrt{3}$.

Hence $\{\frac{v_1}{||v_1|} = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \frac{v_2}{||v_2|} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{v_3}{||v_3|} = \frac{1}{2\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}\}$ is an orthonormal basis for $Col(A)$.

2^0 To find $dist(y, Col(A))$, we need to compute $Proj_{Col(A)}(y)$ first. Since $\{v_1, v_2, v_3\}$ is an orthogonal basis, we have

$$Proj_{Col(A)}(y) = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3.$$

Compute $y \cdot v_1 = \begin{bmatrix} -10 \\ 7 \\ -4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = 10 + 21 - 4 + 3 = 30$, $y \cdot v_2 = \begin{bmatrix} -10 \\ 7 \\ -4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = -30 + 7 - 4 - 3 = -30$ and $y \cdot v_3 = \begin{bmatrix} -10 \\ 7 \\ -4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} = 10 - 7 - 12 - 3 = -12$. We also have $= v_1 \cdot v_1 = 12$, $v_2 \cdot v_2 = 12$ and $v_3 \cdot v_3 = 12$.

Thus

$$\begin{aligned} & Proj_{Col(A)}(y) \\ = & \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{y \cdot v_3}{v_3 \cdot v_3} v_3 \\ = & \frac{30}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} + \frac{-30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} + \frac{-12}{12} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \\ = & \frac{5}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \\ = & \begin{bmatrix} -9 \\ 6 \\ -3 \\ 6 \end{bmatrix}. \end{aligned} \tag{0.2}$$

So $y - Proj_{Col(A)}(y) = \begin{bmatrix} -10 \\ 7 \\ -4 \\ 3 \end{bmatrix} - \begin{bmatrix} -9 \\ 6 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ -3 \end{bmatrix}$ and $dist(y, Col(A)) = ||y - Proj_{Col(A)}(y)|| = \sqrt{12}$.